Incompressible Slip Flows With Permeable Boundaries

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Incompressible Slip Flows With Permeable Boundaries

A Thesis Submitted
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By

S V SACHIDANANDA

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CERTIFICATE

This is to certify that the work embodied in the thesis "Incompressible Slip Flows with Permeable Boundaries" by S V SACHIDANANDA has been carried under my supervision and has not been submitted elsewhere for a degree

Kanpur -
September-1975

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CONTENTS

SYNOPSIS

A Thesis Entitled "Incompressible Slip Flows with Permeable Boundaries" submitted in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy by S V SACHIDANANDA to the Department of Mathematics, Indian Institute of Technology Kanpur, August 1975 under the Supervision of Dr R K Jain

Incompressible flow of fluids over porous boundaries have attained a new dimension with the works of Beavers and Joseph Before their work on flows over porous boundaries, research workers tacitly assumed that there was no relative motion of the fluid with reference to the boundary The purpose of having a porous boundary was either to have a controlled injection of fluid or suction of fluid Beavers and Joseph (1967) proposed an adhoc boundary condition at a permeable wall a linear relation between the velocity and its gradient at a permeable boundary Theoretical predictions on flow rates based on this model were experimentally confirmed Theoretical justifications of the model were advanced by Taylor (1971) and Saffman (1971) and subsequent experimental work by other research workers have confirmed these observations

For flows in ducts and channels with permeable walls, there are some natural queries which comes to one's mind For instance, will the slip condition in the two regions - entry and fully developed region - be the same? How entry length will be affected

and what will be the flow in the entry region ? The present thesis addresses itself to some of these problems

An adhoc variant to the condition of Beavers and Joseph is proposed indicating a variable slip from the leading edge of a permeable flat plate As a consequence slip at the wall is a point function in entry region of the channel This variant has an alternative theoretical basis A search for similar solutions over a permeable flat plate necessitates slip to vary directly as the square root of the distance from the leading edge

This thesis is divided into six chapters Chapter I gives motivation and description of the proposed study A brief review of work done in related topics relevant to the discussion in the thesis is given in Chapter II

Incompressible flow of a viscous fluid over a permeable flat plate is studied in Chapter III As in the Blasius problem, a search for similar solutions is made This requires that the slip velocity at the porous wall be proportional to the square root of the distance from the leading edge along the plate, the proportionality constant depending on the permeability of the plate The method of solution adopted is the numerical integration The velocity profiles are determined for different values of a certain non-dimensional number which depends on the properties of the porous material These profiles lie between the Blasius profile and

the curve $f' = 1$, where f' is the non-dimensional velocity It is observed that the velocity profiles are sharper than the Blasius profile Another method of solution of the same problem is also suggested Under simple transformation of the dependent and independent variables, the governing differential equation is reduced to another boundary value problem which is easier to handle In this formulation one need not take recourse to a method of searching the initial conditions as for example is done for the Blasius problem itself

Chapter IV contains a discussion of the entry flow of an incompressible viscous fluid in a porous walled channel The governing equation of motion are the Prandtl boundary layer equations, and here again allowance for slip at the wall is made The method of solution is the method of von Kármán-Pohlhausen The resulting differential equation has been solved for the two cases when the velocity profiles in the entry region of the channel are similar and are non-similar. The entry length corresponding to two definitions are determined - one through the boundary layer and the other through the fully developed velocity profile In the former case, the entry length is that length when the boundary layer attains half the width of the channel In the latter it is that length when the velocity in the channel attains the value of the velocity of the fully developed flow at the centre of the channel

For both the cases, the entry length is shown to reduce with a non-dimensional parameter which is proportional to the permeability

Chapter V has a discussion of the Bénard problem of a differentially heated permeable block The problem here is to determine the linear instability limit as a function of the properties of the porous material The normal mode analysis has been made and the resulting system of equations have been solved using the Galerkin method The critical Rayleigh numbers so determined are considerably reduced for the porous case as compared with impermeable walled case

The final chapter gives some comments on the thesis content

CHAPTER - 1

INTRODUCTION

The importance of using porous materials as bearing components has been realised because of the fact that the porosity increases the stiffness of the bearing   It also makes the bearing more stable in the dynamic situation   The various bearing configurations of porous metal bearings have been studied by Cameron, Morgan and Stainsby (1962), Joseph and Tao (1966), Shir and Joseph (1966), H Wu (1972) by using both no-slip and slip boundary conditions at the porous surface   The behavior of non-Newtonian fluids in a porous matrix of a bearing has also been recently studied by Isa (1973)   In most of these studies, it has been pointed out that the load bearing capacity decreases with the permeability of the material and with the slip coefficients   But use of the permeable material increases the stiffness

It has been an usual practice to assume that there is no slip at the surface of a permeable material, when there is a flow over it But a recent experimental investigation of Beavers and Joseph (1967) shows that such an assumption is at best a crude approximation to the reality   This is because of the observed migration of fluid tangent to the boundary within the porous material   In order to account for this Beavers and Joseph proposed a model wherein there

is slip at the surface They postulate that the slip velocity at the permeable interface differs from the mean filter velocity within the permeable material and that shear effects are transmitted into the body of the material through a boundary layer region Infact the 'adhoc' condition of Beavers and Joseph implies that the slip velocity at the interface is proportional to the shear at the wall They argue that the proportionality factor depends only on the structure of the material within the porous matrix and not on the flow Subsequent experiments of Sparrow, Beavers, Chen and Lloyd (1972), Beavers, Sparrow and Magnuson (1970) , Taylor (1971) for liquids in Poiseuille flows and Beavers, Sparrow and Masha (1974) for gas flows substantiate the claim of Beavers and Joseph

The above mentioned results are for developed flows A natural question would arise as to what should be the condition at the surface of a porous material for a developing flow, as for example the entry flow in a porous walled channel One anticipates an explicit dependence on a length scale in the direction of the flow, the distance from the leading edge playing this role for the example mentioned above This is because one expects the migration of fluid tangent to the boundary to vary from the leading edge as the outer flow will drag more and more fluid inside the matrix of the porous block as the flow advances in the downstream direction The main body of this thesis addresses itself to this question A mathematical nature of the solutions of the problem discussed then would result in an explicit formulation of the condition

CHAPTER - 2

A REVIEW OF THE LITERATURE

2 1 Permeable walls with no slip

Flow of an incompressible viscous fluid over a porous material has been the subject matter of several research workers The importance of flows with permeable boundaries seems to have originated, specially for the aerodynamical purposes, by Ludwig Prandtl (1904) An important consideration of a flow over an aerofoil is to determine the point of seperation of the flow as the resulting drag thereon is considerable In order to be able to reduce the drag at high speeds, it then becomes necessary to deploy some methods which could delay separation Prandtl was able to show that this could be achieved to a fairly good extent by using slots in the body and by 'sucking in' the retarted fluid in the boundary layer thus making the boundary layer stick close to the body to quite a distance towards the trailing edge A natural outcome of such a study was to introduce a permeable wall instead of having slots in the body Braslow, Burrows, Tetervin and Visconti (1951) have given experimental and theoretical results for a flow over an aerofoil with continuous suction for a considerably high Reynolds number of the order of 10^7 They use a porous aerofoil to achieve the uniform suction Schlichting(1955) and Rosenhead (1963) give a review of the work done in this direction

It is to be observed at this stage that most of the models dealing with the flow of a fluid over a porous body take for the boundary condition at the wall no-slip and prescribe a suction or injection at the wall But when one considers the rarefied flow, one will have to give allowance for the slip at the wall

The effect of wall porosity on the velocity and the pressure distribution in a porous walled channel has been studied by Berman (1953) Using a perturbation procedure he is able to deduce that the velocity profiles are flatter at the center of the channel and steeper at the walls as compared to the Poiseuille flow Yuan (1956) gives a further treatment of the above problem for fairly moderate suction Reynolds number Horton and Yuan (1964) have solved the entry length problem using Kármán-Pohlhausen method Their results agree fairly well with those of Schlichting (1955) and Bodoia and Osterle (1961) Recently Raithby and Knudsen (1974) have studied the hydrodynamic development with suction and blowing The governing equation is the vorticity equation and they solve this numerically They show that for the case of strong suction, the hydrodynamically developed solutions for the velocity profiles previously published are not normally attained Their experimental results confirm this aspect

2 2 Permeable Walls with Slip

Beavers and Joseph (1967) have given some experimental results for the flow of an incompressible viscous fluid in a channel one of

whose walls is permeable Because of the flow of the fluid over the permeable material, there is a migration of fluid tangent to the boundary within the porous body which would not entail one to use the familiar no-slip condition at the wall By the porous wall one means the smooth geometric surface which contains the outermost perimeters of the surface pores of the porous material Beavers and Joseph propose a model which takes into consideration the migration of fluid inside the matrix of the porous material by stating that the slip velocity at the boundary is proportional to the shear at the boundary, the proportionality factor depending only on the geometric structure and the permeability of the material Taylor (1971) gives a justification for the above model with the help of a simple mathematical model and shows that the slip coefficient is indeed independent of the geometry of the measuring device Richardson (1971) in a companion paper to the one of Taylor, gives further analysis of the model Saffman (1971) has given a theoretical foundation for the model of Beavers and Joseph using a statistical approach He first deduces the Darcy's law for a non-homogeneous porous medium and using an asymptotic analysis he deduces the boundary condition of Beavers and Joseph as a consequence.

Sparrow, Beavers, and Hung (1971) consider channel and Tube flows with surface mass transfer using the boundary condition of

Beavers and Joseph at the wall The analysis takes into consideration injection/suction at the surface The results show that the streamwise pressure gradient may increase or decrease with slip as compared to the case when the wall is impermeable Beavers, Sparrow, and Magnuson (1970) have given some experimental results for parallel flows in a porous walled channel Their results show that the mass flow increases where as friction factor decreases as compared to the solid walled channel flows when slip at the porous wall is taken into consideration Sparrow, Beavers, Chen and Lloyd (1973) consider the breakdown of the laminar flow in permeable walled ducts They have conducted experiments with slip velocities at the walls inaddition to the case of zero slip The instability Reynolds number,for both the theory and experiment lie below that corresponding to the impermeable walled case

The effect of velocity slip on the squeeze film between rectangular plates has been studied by Wu (1972) His analysis shows that the existence of slip velocity will further reduce the load carrying capacity and the response time of the squeeze film

A recent investigation of Neale and Nader (1974) on coupled channel flows with bounding porous walls gives a rigorous physical and mathematical basis for the Beavers and Joseph model They show that the slip coefficient α is equivalent to $\sqrt{\tilde{\mu}/\mu}$ where $\tilde{\mu}$ is the effective viscosity and μ the viscosity of the fluid They show

that the Brinkman's extension to Darcy's law (which takes into account a viscous term in the governing equations) accounts for the existence of a boundary layer region within the porous medium and that this would be an important factor when one considers thin channels Infact, they point out that one need not take a recourse to emperical boundary conditions when one uses the Brinkman's extension to Darcy's law

2 3 Plan of the Thesis

Chapters III, IV and V address themselves to answering aspects of flows over a porous flat plate, the entry flow in a porous walled channel, and the problem of setting up of convection currents in a thin film of fluid over a porous block saturated with that fluid respectively For all these problems the condition at the wall generally corresponds to the slip condition of Beavers and Joseph with a variant The flows considered are all laminar, incompressible and the fluid is viscous

It is preferable to use the variant of Beavers and Joseph model to the one presented by Neale and Nader (1974) for two reasons Firstly though the Brinkman equation has received encouraging theoretical verifications from several authors, no experimental verification seems to have been forthcoming so far Secondly it is possibly more appropriate to study the flow in the porous medium statistically and match at the boundary as definition of point

functions inside the porous medium itself are not rigorous Possibly the best course of study would be that of Beavers and Joseph Their condition essentially shows the interaction of flow in the porous material and the external flow and it could perhaps be well-done using a boundary condition rather than solving the two flows separately and matching them

In chapter III, the flow of an incompressible viscous fluid over a porous infinite flat plate saturated with the working fluid is studied The porous plate is assumed to be both homogeneous and isotropic As the flow is over a porous substance, as observed by Beavers and Joseph (1967), allowance for slip at the surface of the material has to be made of This is due to the fact that the outer flow drags with it the fluid in the porous matrix near the surface As the flow starts from the leading edge, more and more of the fluid in the matrix of the porous medium is affected so that one expects the slip at the wall to change continuously with the distance along the flow direction As in the case of flow over an impermeable plate, one looks for similar solutions and as a consequence one finds that the slip velocity varies with the square root of the distance from the leading edge in addition to the already postulated dependence on the shear at the wall The flow equations have been solved with this condition as the basis and the results show that the full velocity is attained faster as contrasted with the impermeable plate case

In chapter IV, the entry flow of an incompressible viscous fluid in a porous walled channel is studied The previous problem becomes a motivation for this case in this sense whether the growth of the thickness of boundary layer increases or decreases as compared with the case of impermeable walled channel One of the boundary conditions used at the wall is similar to the one that is used in the previous problem The method of solution consists of using the von-Kármán-Pohlhausen's approach The entry length determined for this problem is shown to decrease with an increasing value of a parameter which is proportional to the permeability of the porous medium used

In chapter V, the Bénard problem over a porous block is studied The model consists of a porous block saturated with the working fluid on top of which there is a thin layer of this fluid The bottom of this system is heated so that a differential temperature is maintained across the system Because of the interaction between the gravity forces and the viscous forces an instability sets up in the originally static set up and the aim of this investigation is to determine when this happens The results show that the critical Rayleigh numbers are considerably reduced as compared with the case of an impermeable block

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CHAPTER - 3

LAMINAR FLOW OVER A POROUS FLAT PLATE

3 1 Introduction

The problem of the laminar flow of an incompressible viscous fluid over an impermeable flat plate was first studied by Blasius (1908) Using a non-dimensional stream function he reduced the boundary layer equations to an ordinary non-linear differential equation and gave its solution using a series method Subsequently, various methods have been devised by several workers to get a fuller picture of the flow over a flat plate and their results are to be found in Schlichting (~~1962~~ 1955) and Rosennead (1963)

One of the subsequent developments in this direction has been to study the flow of a fluid over a porous flat plate As the plate is porous, fluid can be injected into the external flow or sucked from it The chief reason for studying such a problem, especially the problem with suction, is that suction causes separation to be delayed and this inturn will have a stabilizing effect on the laminar flow As such it has been customary to associate injection or suction with porous materials prescribing these at the wall, the usual no-slip condition at the wall being assumed to hold

Recently Beavers and Joseph (1967) have proposed an emperical relation for the velocity of a fluid at the boundary for a flow over a permeable body They consider the rectilinear flow of a viscous fluid through a two dimensional parallel channel formed by an impermeable upper wall and a permeable lower wall. The flow is generated over and inside the porous body by the application of a pressure gradient maintained in the longitudinal direction Because of the flow over the permeable wall, there is a migration of fluid tangent to the boundary within the porous body But it has been customary, prior to the work of Beavers and Joseph (1967), to disregard this aspect and use the no-slip condition at the surface The no-slip condition in such a situation would be the following the tangential velocity is zero at the wall whereas the normal component is prescribed Beavers and Joseph, therefore, postulate that the slip velocity at the interface differs from the mean filter velocity within the porous material and that shear effects are transmitted into the body of the material through a boundary layer region They assume that the slip velocity for the free fluid is proportional to the shear rate at the boundary, the proportionality constant depending on the geometric structure and the permeability of the porous body They have tested their model and the experimental results are satisfactory

Taylor (1971), Richardson (1971) and Saffman (1971) have given theoretical justifications of the Beavers and Joseph model Taylor gives a simple mathematical model based on the "paint brush model" of the above problem He justifies the claim by Beavers and Joseph that the slip coefficient need only depend on the structure of the porous material, out not on the features of the geometry of the measuring device Richardson in a companion paper gives further analysis of the Taylor model Saffman gives a justification of the proposed condition employing a statistical approach He first deduces the Darcy's law for a non-homogeneous porous medium and using an asymptotic analysis he deduces the boundary condition as a consequence

In the present investigation, the flow of an incompressible viscous fluid over an infinite porous flat plate is restudied As the plate is porous, the no-slip condition is discarded Instead, a boundary condition similar to that of Beavers and Joseph is assumed The method of solution is the numerical solution

3 2 The Model

The physical problem is the laminar flow of an incompressible viscous fluid over a flat porous plate The plate is assumed to be both homogeneous and isotropic The plate is saturated with

the fluid before the flow starts The x-coordinate is taken along the plate and the y-coordinate normal to it with the leading edge of the plate being taken as the origin A sketch of the model is as shown in Fig 3 1 There is a uniform flow of the fluid with a velocity U_∞ upstream of the plate The problem is to know how the velocity developes on the plate

The governing differential equations are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (3\ 1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3\ 2)$$

where ν is the kinematic viscosity of the fluid The boundary conditions at the surface are

$$\phi\left(\frac{xU_\infty}{\nu}\right) \frac{\partial u}{\partial y} = \frac{\alpha}{\sqrt{\kappa}} u \ , \ v = 0 \quad \text{at} \quad y = 0 \qquad (3\ 3)$$

$$u \to U_\infty \quad \text{as} \quad y \to \infty \qquad (3\ 4)$$

Here α is the slip coefficient which depends on the geometry of the material and κ is its permeability κ is a measure of the fluid conductivity of the material

Some comments about the boundary conditions as to their forms are relevant here The first condition corresponds to the

slip condition of Beavers and Joseph, and as in their paper is taken to be an emperical relation. When the full velocity profile is attained, which in practice is attained for a finite distance x, the resulting condition at this station and onwards reduces to the condition of Beavers and Joseph. The adhoc condition assumes that the slip is a function of x, the distance from the leading edge, till the full profile is attained. As there is no preferred length for the model, $\left(\frac{U}{\nu}\right)^{-1}$ has been taken as a length scale.

As in the case of Blasius problem, when one searches for similar profiles for the flow, the function ϕ becomes $\sqrt{\frac{xU_\infty}{\nu}}$. The second condition in Equation (3.3) shows that there is no suction or injection. This condition differs from the usual analyses of flows over a porous body wherein v is prescribed.*

Introducing a non-dimensional stream-function $f(\eta)$ and the transformations

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \quad , \quad \psi = \sqrt{\nu x U_\infty}\, f(\eta)$$

Equations (3.1) and (3.2) reduce to

* Infact, under the Prandtl approximation, when an order of magnitude analysis is made, it can be shown that v is of the order of δ. As such it has been assumed to be zero.

$$2f''' + ff'' = 0 \qquad (3\ 5)$$

where primes denote differentiation with respect to η

As similar solutions are sought for, the boundary conditions become

$$f(0) = 0,\ f''(0) = \beta f'(0)\ ,\ f' \to 1 \quad \text{as} \quad \eta \to \infty \qquad (3\ 6)$$

where use has been made that the function ϕ can be only $\sqrt{\frac{xU_\infty}{\nu}}$ and $\beta = \frac{\nu\alpha}{U_\infty\sqrt{K}}$

3 3 Solution

The method of solution is the numerical integration The differential equation (3 5) has been integrated using the Runge-Kutta method (Carnahan, Luther and Wilkes (1969)) with the relevant boundary conditions An alternate method for solving the Equation (3 5) is also given

The differential Equation (3 5) can be written as a set of three ordinary differential equations

$$\begin{aligned} \frac{dg_1}{d\eta} &= g_2 \\ \frac{dg_2}{d\eta} &= g_3 \qquad (3\ 7) \\ \frac{dg_3}{d\eta} &= -\frac{1}{2}\, g_1 g_3 \end{aligned}$$

The boundary conditions become

$$g_1(0) = 0, \; g_3(0) = \beta g_2(0), \; g_2 \to 1 \quad \text{as} \quad \eta \to \infty \qquad (3\ 8)$$

The above system can be treated as 'initial' value problems for a given initial condition $g_2(0)$, as two of the initial conditions could then be had Then a search for a value of g_2 at $\eta = 0$ is made that will generate a solution for which $g_2 \to 1$ as $\eta \to \infty$ This can be accomplished by having an iterative procedure for $g_2(0)$ Then the whole system of equations can be solved using the Runge-Kutta method

The range of values of β is from 0 to ∞ , the former value corresponding to the infinite permeability case-hence an infinite slip- which is an 'ideal' situation and the latter case corresponds the Blasius problem Thus the velocity curves are to lie between the Blasius profile and the 'ideal' profile $f' \equiv 1$

The velocity profiles for some representative values of β are drawn and are shown in Fig 3 2

An alternative approach for solving the Equations (3 5) and (3 6) can be given The advantage of this method is that the given boundary value problem is reduced to another boundary value problem wherein a search for the initial condition need not be made

Making the transformations

$$\eta^* = \eta\beta \quad \text{and} \quad f = 2\,\beta g \tag{3 9}$$

the given boundary value problem reduces to

$$g''' + gg'' = 0 \tag{3 10}$$

With the boundary conditions

$$g(0) = 0, \; g''(0) = g'(0), \; g' \to \frac{1}{2\beta^2} \text{ as } \eta^* \to \infty \tag{3 11}$$

where primes now denote differentiation with respect to η^*

The above problem could be solved as in the previous problem with this difference The integration can be done starting with some initial condition, the integration being done till g' attains a constant value which will give the value of β A sketch of the profiles g' against η^* is given in Fig 3 3

3 4 Conclusions

The velocity profiles for different values of the parameter β are shown in Fig 3 2 The Blasius profile corresponds to the value β 'equal' to infinity It is to be observed that the profiles for increasing values of β are one below the other The curves for smaller values of β have

sharper curvatures and on already observed, the ideal situation of infinite permeability corresponds to the curve $f' \equiv 1$ This suggests that the full velocity is attained faster as compared to the Blasius profile

It is to be noted that the usual compatibility conditions are not available for the above model because of the allowance for the slip at the wall and the absence of any suction or injection at the wall As such no quantitative comparisons can be made as the problems are physically different But some remarks could be made about the similarities between the model considered and the model describing the flow over a porous flat plate with suction without slip The asymptotic suction profile is sharper as contrasted with the Blasius profile (Schlichting, pp 270) For the problem under consideration a similar result is true The measurements of Head (1951) seems to suggest similar results as he has made measurements for relatively small suction and his results being true from a short distances away from the leading edge

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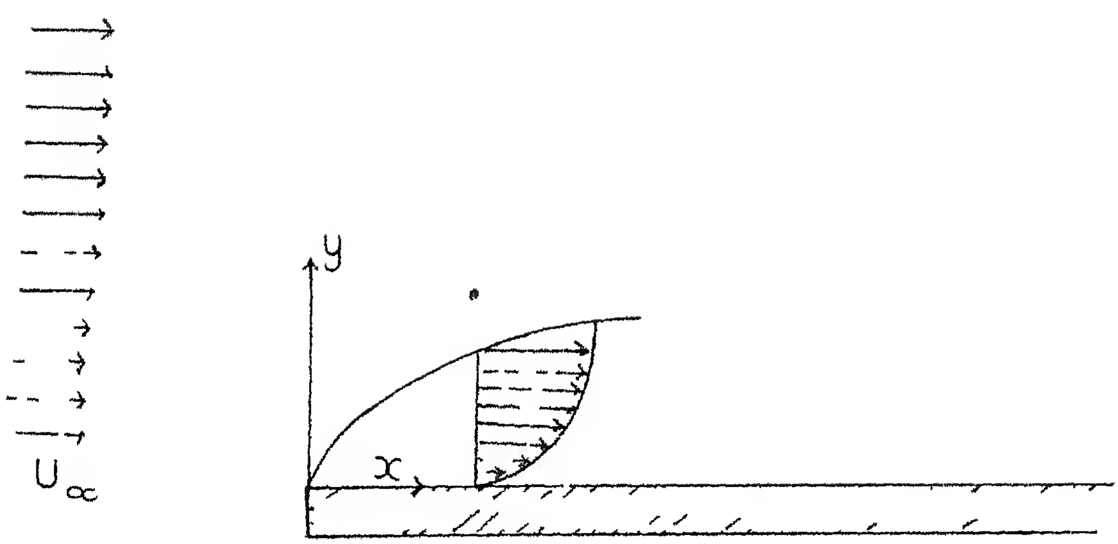


FIG 31 BOUNDARY LAYER ALONG A POROUS FLAT PLATE

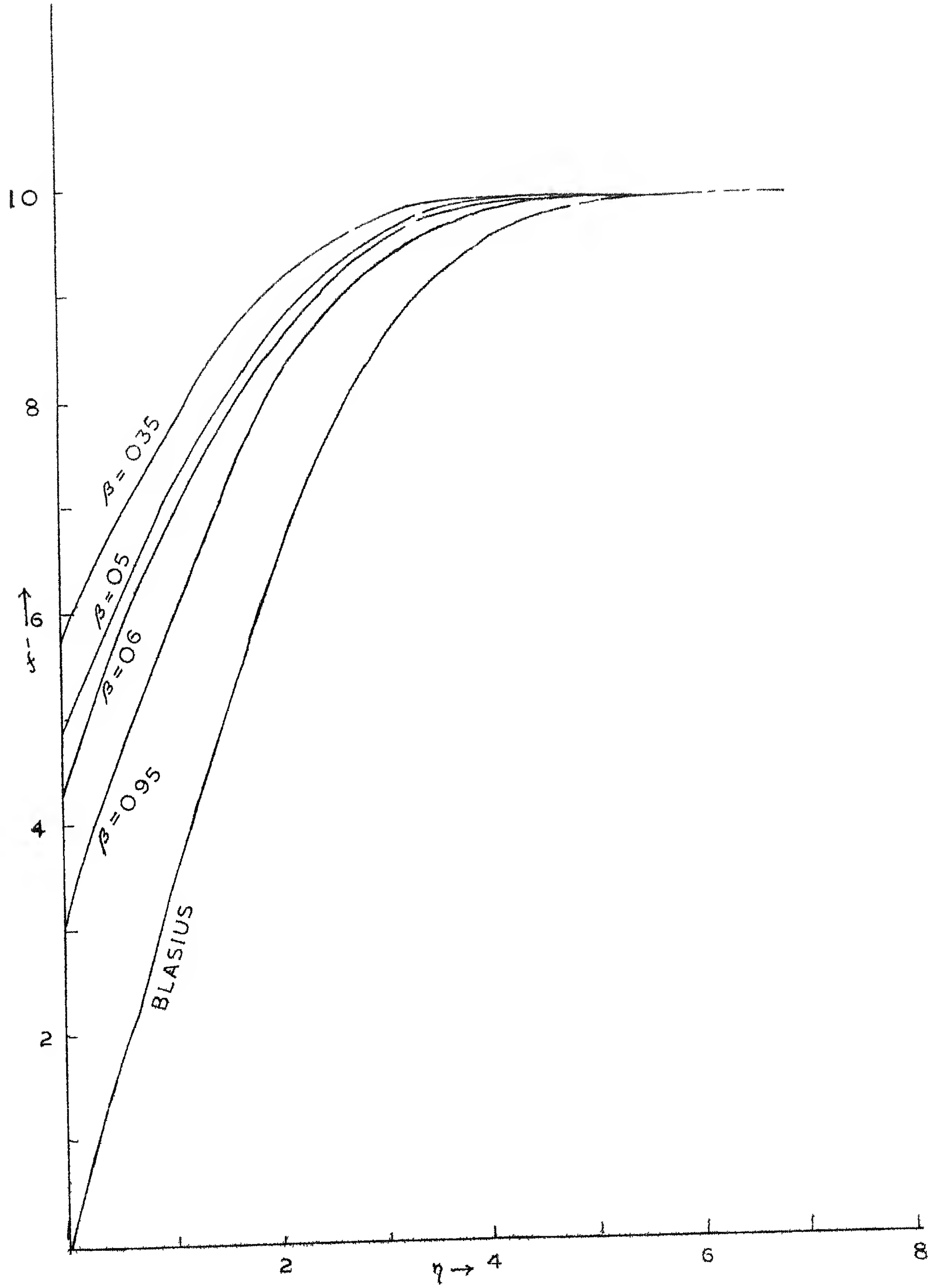
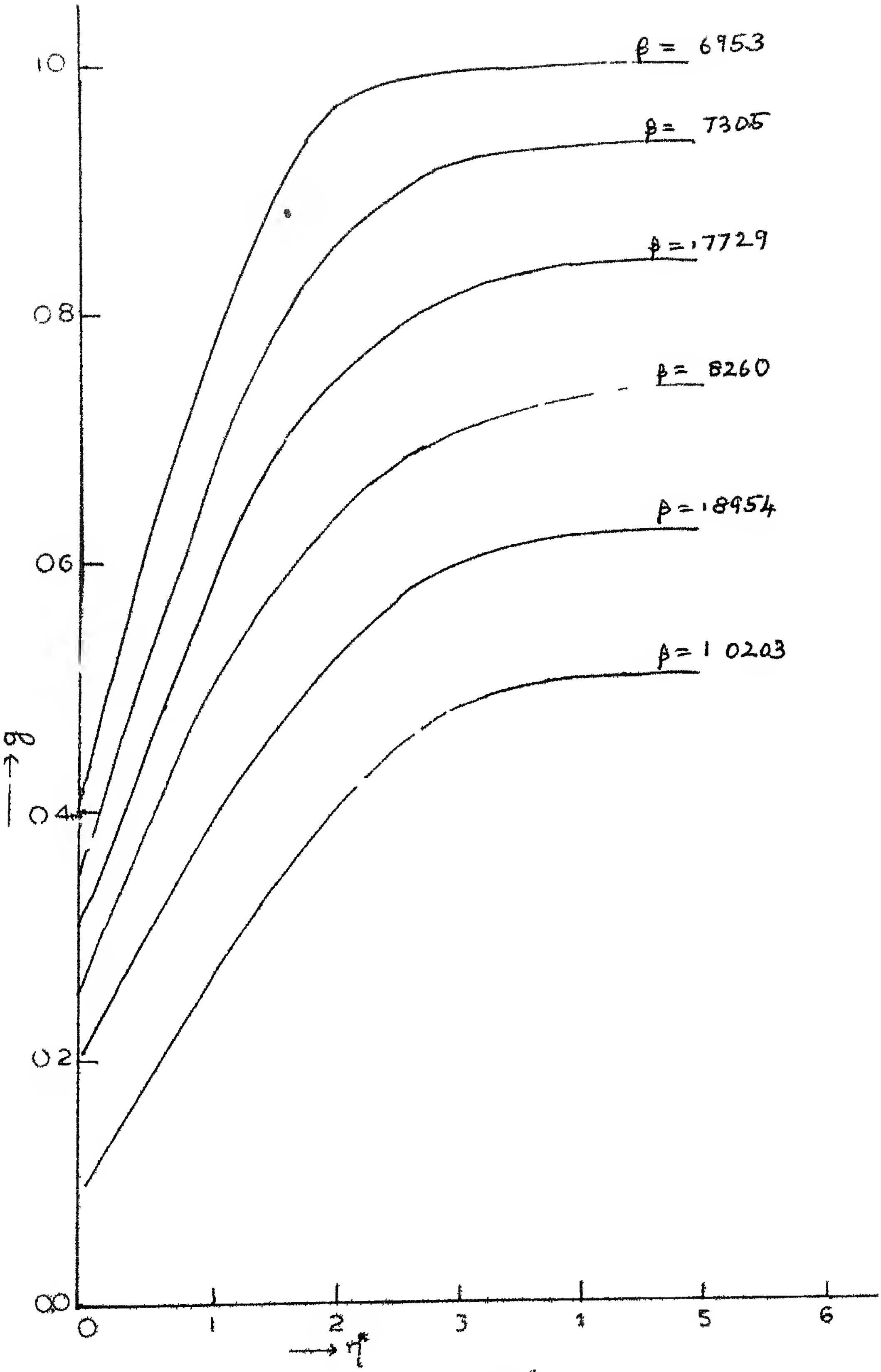


FIG 32 VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ALONG A POROUS FLAT PLATE



β = 6953
β = 7305
β = ·7729
β = 8260
β = ·8954
β = 1 0203
10
08
06
04
02
00
g
0
1
2
3
4
5
6
η*

FIG 33 g′ VS η*

CHAPTER - 4

ENTRY FLOW IN A POROUS WALLED CHANNEL

4 1 Introduction

The flow of an incompressible viscous fluid in a channel can generally be divided into two regions an entrance region which is characterized by the changing of velocity profiles continuously in the direction of the flow and a fully developed region wherein velocity profiles exist independent of the flow direction

The steady laminar flow of an incompressible Newtonian fluid in the entrance region or the inlet section of a channel has been investigated by Schlichting (1955), Bodoia and Osterle (1961), Collins, Morton and Schowalter (1962) and others These authors use the usual boundary layer theory In all these cases the velocity distribution has been assumed to be flat at the entrance of the channel and the boundary layer approximation of neglecting the viscous term $\mu \frac{\partial^2 u}{\partial x^2}$ and the pressure gradient normal to the flow direction has been made Schlichting (1955) using the original

Blasius (1908) technique for a flat plate obtained the velocity profile for the entrance region by means of a matching procedure between an upstream and a downstream solution to the differential equation governing the system Bodoia and Osterle (1961) solved the entry problem by means of a finite difference procedure and obtained results which differed from Schlichting's solution They explained the discrepancy as follows firstly there is always a discontinuity introduced at the point where the upstream and downstream solutions were matched and secondly Schlichting's assumption of the second derivative of the forward velocity with respect to the cross channel position variable being zero is incorrect Collins and Schowalter (1962) used Schlichting's method with refinements Recently van Dyke (1970) and S D R Wilson (1971) have worked on this problem of development of a channel flow using essentially the same techniques

The effect of wall porosity on the velocity and the pressure distribution in a porous walled channel has been investigated possibly for the first time by Berman (1953) As he studies the fully developed two-dimensional flow, he uses a stream function to reduce the full Navier-Stokes equations to a third order non-linear differential equation with the usual boundary conditions with a prescribed constant suction at the wall Using a perturbation procedure he is able to deduce that the velocity profiles are

flatter at the centre of the channel and steeper at the walls as compared to the Poiseuille flow and that the pressure drop in the direction of the flow is found to be appreciably less in the porous walled channel corresponding to an impermeable walled channel of the same dimensions and the same entry Raynolds number Yuan (1956) gives a further treatment of the above problem when the suction Reynolds number is fairly moderate Horton and Yuan (1964) have solved the entry flow problem using the Kármán-Pohlhansen method They give solutions for both similar as well as non-similar profiles in the entry region The parameter used in the calculation is a Reynolds number depending on the suction velocity at the wall Their results agree fairly well with those of Schlichting (1955) and Bodoia and Osterle (1961)

Sparrow, Beavers and Hung (1971) have studied the problem of channel and tube flows with surface mass transfer and velocity slip at wall They use the Beavers and Joseph model for the boundary condition at the wall Their analysis shows some interesting results because of the presence of slip at the wall They show that relative to the situation where there is no slip, the streamwise pressure gradient may either increase or decrease in the presence of slip and may even undergo a change of sign

However, the entry flow in channels with permeable walls allowing slip does not seem to have been studied so far

In the present investigation, the entry flow of an incompressible viscous fluid in a permeable walled channel is studied. The permeable walls are assumed to be homogeneous and isotropic and they are saturated with the working fluid to start with. Generally, two types of flow at the entry are assumed: (i) the flow is uniform at the entrance of the channel and (ii) the flow upstream of the leading edge of one of the walls is assumed to be uniform. Assumption (i) is used more frequently though the second assumption is more realistic physically. Wang and Longwell (1964) have shown that very definite effects are transmitted upstream from the leading edge of the plates at a Reynolds number of 300 for the first type of initial condition. They have also found that the overall pressure drop and the inlet length predicted by boundary layer theory methods are in reasonable agreement with those of the boundary conditions corresponding to the case (ii). They feel that this might be due to compensating errors caused by the several assumptions made in the boundary layer theory. van Dyke, M (1970) has observed that use of condition (i) introduces a weak vorticity at the inlet and that this leads to certain complications.

In the present analysis the flow is assumed to be uniform at the entry. The method of solution adopted is based on the van Kármán-Pohlhausen method. Briefly the method is as follows

The boundary layer equations are integrated once to obtain the Kármán momentum integral equation A form of velocity profile u(x,y) is then sought which satisfies the momentum equation and the relevant boundary conditions When the assumed velocity profile is substituted in the momentum integral equation, these results a differential equation for the shape factor which is a function of the external pressure gradient and the boundary layer thickness By solving for the boundary layer thickness, the other variable like the displacement thickness, the momentum thickness are determined Pohlhausen used a quartic expansion for the velocity profile

Several improvements of the Kármán-Pohlhausen method are available Mangler (1944) suggested the use of a polynomial of the n-th degree for the velocity profile which automatically satisfies some of the boundary conditions Timman (1949) has suggested the use of certain indefinite integrals for the velocity profile Several other 'approximate' methods are available which use in addition to the momentum integral equation, the energy integral equation also All these methods work fairly well when the flows are accelerating - the flow developing in a channel is an example of such a flow A fairly exhaustive treatment of these approximate methods is available in Rosenhead (1963) and Curle (1962)

4 2 Model and Mathematical Formulation

The model consists of a two dimensional steady flow of an incompressible viscous fluid in a permeable walled channel The walls are 2h units apart and are assumed to be both homogeneous and isotropic The walls of the channel are assumed to be saturated with the working fluid to begin with The permeability of the channel walls is κ Because of the symmetry in the flow field, only the flow in one half of the channel is studied A coordinate frame is fixed at the leading edge, it being taken as the origin The x-axis is along the flow direction along the channel and the y-axis is in a transverse direction(see Fig 5 1)

The governing equations of motion are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial}{\partial x} (p/\rho) + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{4 1}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial}{\partial y} (p/\rho) + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{4 2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4 3}$$

The boundary conditions are

$$\text{at} \quad y = 0 \quad u = \frac{\sqrt{\kappa}}{\alpha} f(x) \frac{\partial u}{\partial y}, \; v = 0 \tag{4 4}$$

$$y = \delta \qquad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \qquad = 0 \ , \ u = f \tag{4 5}$$

Here U is the central line velocity, κ is the permeability of the porous walls, α is the slip coefficient and $\delta = \delta(x)$ is the local boundary layer thickness

Let $\bar{u} = \frac{u}{U_\infty}$, $\bar{v} = \frac{vh}{U_\infty \delta}$, $\bar{p} = \frac{p}{\rho U_\infty^2}$, $\bar{U} = \frac{U}{U_\infty}$, $\bar{x} = \frac{x}{h}$, $\eta = \frac{y}{\delta}$,

$Pe = \frac{U_\infty h}{\nu}$, where U_∞ is the constant external velocity, 2h is the width of the channel

Then making the Prandtl approximations, these equations (4 1), (4 2) and (4 3) become (after dropping bars)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial \eta} = U \frac{dU}{dx} + \frac{1}{Re} \frac{h^2}{\delta^2} \frac{\partial^2 u}{\partial \eta^2} \tag{4 6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial \eta} = 0 \tag{4 7}$$

The non-dimensional boundary conditions are

$$\text{at} \quad \eta = 0 \qquad u = \frac{\sqrt{\kappa}}{\alpha} \frac{f(\bar{x})}{\delta} \frac{\partial u}{\partial \eta} \ , \ v = 0 \tag{4 8}$$

$$\eta = 1 \qquad \frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} = \qquad = 0, \ u = U \tag{4 9}$$

Integration of the equation (5 6) from $\eta = 0$ to $\eta = 1$ and use of continuity equation gives

$$\frac{1}{Re} \frac{h^2}{\delta^2} \left.\frac{\partial u}{\partial \eta}\right|_{\eta=0} = \frac{d}{dx} \int_0^1 u(U-u)d\eta + \frac{dU}{dx} \int_0^1 (U-u)d\eta \qquad (4\ 10)$$

It is the well known Kármán's integral equation

4 3 Solution

The method of solution adopted is similar to the von-Kármán-Pohlhausen method The velocity in the boundary layer is expanded as a quartic in the non-dimensional transverse variable η The coefficients are determined by using sufficient number of boundary conditions from equations (4 9) Let then the velocity profiles be given by

$$\frac{u}{U} = \sum_0^n A_i \eta^i + \phi(x) \sum_0^n B_i \eta^i , \qquad (4\ 11)$$

where $\phi(x)$ is a function of x alone and A_i and B_i are constants The constants in the above expansion depend upon the degree of the polynomial chosen The form chosen as in Equation (4 11) will be useful later when discussion for the similar andnon-similar profiles is done As for example, in the case of similar profiles $\frac{u}{U}$ has to be a function of η alone so that the second term in the right hand side of Equation (4 11) is dropped for that discussion Discussion for the quartic polynomial alone is done in the subsequent sections

Solution for similar profiles

If the assumption is made that the velocity profiles in the boundary layer are similar at all channel cross-sections, then Equation (4 11) will reduce to $\frac{u}{U} = \sum_{0}^{4} A_i \eta^i$. This also implies in turn that the coefficient $\frac{\sqrt{\kappa}}{\sigma}\ \frac{f(x)}{\delta}$ in Equation (4 8) be a constant say, d. The coefficients A_i's have to satisfy the Equations (4 8) and (4 9) when

$$A_0 + A_1 + A_2 + A_3 + A_4 = 1$$

$$A_1 + 2A_2 + 3A_3 + 4A_4 = 0$$

$$2A_2 + 6A_3 + 12A_4 = 0$$

$$6A_3 + 12A_4 = 0$$

$$A_0 = dA_1$$

Thus

$$A_0 = \frac{4d}{1 + 4d}, \quad A_1 = \frac{4}{1 + 4d}, \quad A_2 = -\frac{6}{1 + 4d}$$

$$A_3 = \frac{4}{1 + 4d} \quad \text{and} \quad A_4 = -\frac{1}{1 + 4d}$$

Hence

$$\frac{u}{U} = \frac{1}{(1 + 4d)} \left[(1 + 4d) - (1 - \eta)^4 \right] \tag{4 12}$$

Also,

$$\int_0^1 u\,d\eta = \frac{20d + 4}{5(1 + 4d)}\,U \tag{4 13}$$

$$\int_0^1 u^2\,d\eta = \frac{(720d^2 + 288d + 32)}{45\,(1 + 4d)^2}\,U^2 \tag{4 14}$$

The central line velocity U which is a function of x alone needs to be determined This can be done since a definite form of expression for u has already been assumed satisfying the boundary conditions Also at two different sections of the channel, the inflow across one section must be equal to the outflow across the other section of the channel So much so, the inflow across the section at the entrance must be equal to the outflow across any section of the channel The flow inside the channel itself could be divided into two halves-flow in the boundary layer where the velocity component is u and the flow outside the boundary layer, the accelerating core, where U is the velocity

Thus writing the mass conservation in a dimensional form one gets

$$\int_0^{2h} u\,dy = \int_0^{2h} U_\infty\,dy \tag{4 15}$$

or non-dimensionally (after dropping the bars)

$$\delta \int_0^1 u \, d\eta + \delta \int_1^{h/\delta} U \, d\eta = h \qquad (4\ 16)$$

Substituting in Equation (4 16) for $\int_0^1 u \; d\eta$ from Equation (4 13) and simplifying one obtains

$$U = \frac{1}{1 + B\frac{\delta}{h}} , \qquad (4\ 17)$$

where

$$B = - \frac{1}{5(1 + 4d)}$$

Thus the momentum integral equation becomes

$$\frac{4}{Re} \frac{h^2}{\delta^2} \frac{U}{(1 + 4d)} = U \frac{dU}{dx} \left[\frac{2(20d + 4)}{5(1 + 4d)} - 2 \frac{(720d^2 + 288d + 32)}{45(1 + 4d)^2} \right.$$

$$\left. + 1 - \frac{(20d + 4)}{5(1 + 4d)} \right] = U \frac{dU}{dx} \frac{(108d + 17)}{45(1 + 4d)^2}$$

or

$$\frac{4}{Re} \frac{h^2}{\delta^2} = A \frac{dU}{dx} , \qquad (4\ 18)$$

where $A = \dfrac{108d + 17}{45(1 + 4d)}$

Substituting for $\frac{h}{\delta}$ from Equation (4 17) and integrating from 0 to x,

$$x = \frac{A\,Rc}{4B^2}\left[\frac{U^2-1}{U} - 2\,\ln U\right] \tag{4 19}$$

Working on similar lines for the case of a rigid boundary, that is, when $u = 0$ at $\eta = 0$, one gets

$$x = \frac{85}{36}\,Re\,\left[\frac{U^2-1}{U} - 2\,\ln U\right] \tag{4 20}$$

This equation turns out to be a particular case of Equation (4 19) for $d = 0$

4 4 Non-Similar Profiles

For the velocity profiles in the boundary layer to be non-similar, the velocity profile $\frac{u}{U}$ is as given in Equation (4 11) As only quartic polynomials are being considered

$$\frac{u}{U} = \sum_{o}^{4} A_i\eta^i + \phi(x)\sum_{o}^{4} B_i\eta^i \tag{4 23}$$

Without loss of generality, B_o can be taken to be 1 Some of the constants A_i, B_i can be found as in the previous case Thus using the Equations (4 8) and (4 9) one has

$$A_o = 1 + A_4\ ,\ A_1 = -4A_4 = A_3\ ,\ A_2 = 6A_4$$

$$B_o = 1 = B_4\ ,\ B_1 = -4B_4 = B_3\ ,\ B_2 = 6B_4$$

Equation (4 8) yields

$$A_o + \phi B_o = g(A_1 + \phi B_1)$$

or $$\psi = -\frac{1}{1+4g} \,, \tag{4 24}$$

where $$g = \frac{\sqrt{\kappa}}{\alpha}\frac{f(x)}{\delta} \,, \quad \psi = A_4 + \phi B_4 \tag{4 25}$$

The function $g(x)$ treated as a function of the parameter κ will have the range $[0, \quad)$, the end points corresponding to the rigid and the 'completely' permeable case The latter case is physically non-plausible Thus the range of the function ψ is $[-1, 0)$

Equation (4 23) then reduces to

$$\frac{u}{U} = (1+\psi) - 4\psi\eta + 6\psi\eta^2 - 4\psi\eta^3 + \psi\eta^4$$

$$= 1 + \psi(1-\eta)^4 \tag{4 26}$$

Also

$$\int_0^1 u\, d\eta = U\left(1 + \frac{\psi}{5}\right), \tag{4 27}$$

$$\int_0^1 u^2 d\eta = U^2\left(1 + \frac{\psi^2}{9} + \frac{2}{5}\psi\right) \tag{4 28}$$

From the boundary condition given by Equation (4 8) one has

$$u \frac{\partial u}{\partial x} = U \frac{dU}{dx} + \frac{1}{Re} \frac{h^2}{\delta^2} \frac{\partial^2 u}{\delta \eta^2} \quad \text{at } \eta = 0 \tag{4 29}$$

As in the previous case U is determined by the use of the conservation law

$$\delta \int_0^1 u \, d\eta + \delta \int_1^{h/\delta} U \, d\eta = h \tag{4 30}$$

Substituting for the former integral

$$U = \frac{1}{1 + \frac{\psi}{5} \frac{\delta}{h}} \tag{4 31}$$

Equation (4 29) then gives

$$\frac{12}{Re} \frac{h^2}{\delta^2} = \frac{1}{\psi} \frac{d}{dx} (U\psi) + \frac{dU}{dx} + \frac{d}{dx} (\psi U) \tag{4 32}$$

Substituting for the integrals in the momentum Equation (4 10), one further obtains

$$\frac{4}{Re} \frac{h^2}{\delta^2} = \left(\frac{3}{5} + \frac{2}{9} \psi \right) \frac{dU}{dx} + U \left(\frac{1}{5\psi} \frac{d\psi}{dx} + \frac{2}{9} \frac{d\psi}{dx} \right) \tag{4 33}$$

Eliminating $\frac{1}{Re} \frac{h^2}{\delta^2}$ between the above two equations

$$\frac{dU}{d\psi} = \frac{\frac{dU}{dx}}{\frac{d\psi}{dx}} = -\frac{U}{\psi} \left(\frac{6 + 5\psi}{3 + 5\psi} \right) \tag{4 34}$$

On integration,

$$U = c \frac{(5\psi + 3)}{\psi^2} \qquad (4\ 35)$$

where c is a constant of integration

Equation (4 32) with the help of Equation (4 31) and (4 35) would then become

$$\frac{dx}{d\psi} = -\frac{25\ Re}{12} \frac{(9 + 8\psi)\,[\psi^2 - c\ (5\psi + 3)]^2}{\psi^5\ (3 + 5\psi)^2} \qquad (4\ 36)$$

The above equation can be integrated in a straight forward fashion Before doing so, certain comments are due

The flow in the channel is an accelerating flow so that $\frac{dU}{dx} > 0$ But

$$\frac{dU}{dx} = \frac{dU}{d\psi} \frac{d\psi}{dx} \qquad (4\ 37)$$

Thus $\frac{dU}{d\psi}$ and $\frac{d\psi}{dx}$ need to have the same signs But from Equation (4 36), for the range of ψ which is $[-1,0)$, $\frac{dx}{d\psi} > 0$ So much so, $\frac{dU}{d\psi} > 0$ But

$$\frac{dU}{d\psi} = \frac{-c}{\psi^3} \quad (5\psi + 6) \qquad (4\ 38)$$

Hence c is positive

Also since U is positive, and c is also positive, Equation (4 35) shows that

$$(5\psi + 3) > 0$$

And thus

$$-6 < \psi < 0 \qquad (4\ 39)$$

Integrating Equation (4,36) one gets

$$x = -\frac{25}{12}\,\mathrm{Re}\left[\frac{9-70c}{9}\ \ln\left(\frac{5\psi+3}{\psi}\right) - \frac{9}{4}\ \frac{c^2}{\psi^4} - \frac{8}{3}\ \frac{c^2}{\psi^3}\right.$$

$$\left. + \frac{3c}{\psi^2} - \frac{14c}{3\psi} + \frac{7}{5(3+5\psi)}\right] + A, \qquad (4\ 40)$$

where A is a constant of integration

Since $U = 1$ at $x = 0$, from Equation (4 35) one has

$$\psi_o^2 = c\,(3 + 5\psi_o)\,, \qquad (4\ 41)$$

where ψ_o is the value of ψ at $x = 0$

4 5 Discussion

The entry length for the channel could be defined in two ways (i) through the boundary layer developing in the channel and (ii) through the fully developed velocity profile Considering the former case

Definition I

The entry length for a porous walled channel is that length along the channel when the boundary layer grows asymptotically to half the channel width

By symmetry of the flow in the channel with respect to the center of the channel, the boundary layers developing on the two walls of the channel meet and the distance of this point from the entrance is the required entry length This sort of definition would give a lower bound for the entry length, (see D vies (1972))

But a more appropriate definition would be

Definition II

The entry length for a porous walled channel is that length along the channel attained when the velocity profile of the developing flow in the channel attains the value of the velocity of the fully developed flow t the center of the channel

The procedure for the determination of the entry length for the similar profiles in the boundary layer is as follows when the profiles are similar, the boundary condition given by Equation (4 8) is to reduce to

$$u = d \frac{\partial u}{\partial \eta} \quad \text{at} \quad \eta = 0 \tag{4 42}$$

But for the fully developed case, i e , when the dimensional form of Beavers and Joseph boundary condition is used

$$u = \frac{\sqrt{\kappa}}{\alpha} \frac{\partial u}{\partial y} \quad \text{at } y = 0 \tag{4 43}$$

or in the non-dimensional form

$$u = \frac{\sqrt{\kappa}}{\alpha h} \frac{h}{\delta} \frac{\partial u}{\partial \eta} \quad \text{at } \eta = 0 \tag{4 44}$$

If x_L is the entry length, then

$$d = \frac{\sqrt{\kappa}}{\alpha h} \frac{h}{\delta(x_L)} = K \frac{h}{\delta(x_L)} , \; K = \frac{\sqrt{\kappa}}{\alpha h} \tag{4 45}$$

But then from Equation (4 17),

$$U(x_L) = \frac{1}{1 - \frac{1}{5(1+4d)} \frac{\delta}{h}(x_L)} = \frac{5(1+4d)}{5(1+4d) - \frac{K}{d}} \tag{4 46}$$

Using the boundary condition given by Equation (4 43), one can show that u_{max} , the fully developed value of the centre line velocity is

$$u_{max} = \frac{3}{2} \left(\frac{1 + 2K}{1 + 3K} \right) \tag{4 47}$$

Thus using the Definition II of the entry length, one has

$$d = \frac{-5 + \sqrt{25 + 240K\,(1 + 2K)}}{40} \tag{4 48}$$

The entry length x_L for the Definition I is calculated by taking $\delta/h = 1$ in Equation (4 46) which determines U Then x_L could be determined from Equation (4 19) for different values of d

The entry lengths corresponding to different values of K have been calculated for the above two cases and they are tabulated in Table (4 1)

For the non-similar profiles, a similar approach would give one

$$g(x_L) = \frac{-5 + \sqrt{25 + 240K\,(1 + 2K)}}{40} \tag{4 49}$$

But

$$\psi\,(x_L) = -\frac{1}{1 + 4g(x_L)} \tag{4 50}$$

Again using the Definition II of the entry length,

$$\frac{c\,\psi^2(x_L)}{3 + 5\,\psi(x_L)} = \frac{3}{2}\,\left(\frac{1 + 2K}{1 + 3K}\right) \tag{4 51}$$

Given a K, $\psi\,(x_L)$ could be determined from Equation (4 50) and hence c from Equation (4 51) Then ψ_o the value of $\psi\,(x)$ at $x = 0$ could then be determined from Equation (4 41)

The entry length x_L corresponding to the given K then could be determined from Equation (4 40)

The entry length corresponding to Definition I, i e , in terms of the boundary layer growing to half the channel width is calculated as in the previous case of similar profiles, since one has a relation between U and δ/h from Equation (4 31), a relation between U and ψ from Equation (4 35) and a relation between, c and ψ_0 in Equation (4 41)

A tabulation of entry lengths x_L for different values of K are given in Table (4 2) for the case of both the definition

For both the similar and non-similar velocity profiles in the boundary layer of the developing flow, one notices from the Tables (4 1) and (4 2), that the entry length reduces with the increasing values of the parameter K, i e , with the permeability of the walls for a given value of α , the slip coefficient The chosen values of K are relatively small compared to the range of K which can be $[0, \infty)$ Also one notices that the value of the entry lengths for the definition given interms of the boundary layer attaining half the channel width is less than that of the criterion in terms of the value of the fully developed profile which conforms with the statement of Davies(1972)

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Table 4 1

Entry Length Similar Profile.

K	Entry Length Defn I	Entry Length Defn II
0	$8\ 972 \times 10^{-3}$	$5\ 313 \times 10^{-2}$
0 001	$8\ 750 \times 10^{-3}$	$5\ 283 \times 10^{-2}$
0 002	$8\ 734 \times 10^{-3}$	$5\ 277 \times 10^{-2}$
0 003	$8\ 718 \times 10^{-3}$	$5\ 270 \times 10^{-2}$
0 004	$8\ 712 \times 10^{-3}$	$5\ 263 \times 10^{-2}$
0 005	$8\ 685 \times 10^{-3}$	$5\ 257 \times 10^{-2}$
0 006	$8\ 669 \times 10^{-3}$	$5\ 250 \times 10^{-2}$
0 007	$8\ 653 \times 10^{-3}$	$5\ 244 \times 10^{-2}$
0 008	$8\ 637 \times 10^{-3}$	$5\ 237 \times 10^{-2}$
0 009	$8\ 622 \times 10^{-3}$	$5\ 230 \times 10^{-2}$
0 010	$8\ 606 \times 10^{-3}$	$5\ 224 \times 10^{-2}$
0 015	$8\ 527 \times 10^{-3}$	$5\ 190 \times 10^{-2}$
0 017	$8\ 496 \times 10^{-3}$	$5\ 176 \times 10^{-2}$
0 018	$8\ 481 \times 10^{-3}$	$5\ 170 \times 10^{-2}$
0 019	$8\ 466 \times 10^{-3}$	$5\ 163 \times 10^{-2}$

Table 4 2

Entry Length Non-Similar Profiles

K	Entry Length	
	Defn I	Defn II
0 35	$8\ 435 \times 10^{-3}$	$5\ 872 \times 10^{-2}$
0 40	$3\ 956 \times 10^{-3}$	$2\ 734 \times 10^{-2}$
0 45	$2\ 387 \times 10^{-3}$	$1\ 637 \times 10^{-2}$
0 50	$1\ 615 \times 10^{-3}$	$1\ 099 \times 10^{-2}$
0 55	$1\ 164 \times 10^{-3}$	$7\ 885 \times 10^{-3}$
0 60	$8\ 792 \times 10^{-4}$	$5\ 917 \times 10^{-3}$
0 65	$6\ 850 \times 10^{-4}$	$4\ 588 \times 10^{-3}$
0 70	$5\ 463 \times 10^{-4}$	$3\ 647 \times 10^{-3}$
0 75	$4\ 448 \times 10^{-4}$	$2\ 951 \times 10^{-3}$
0 80	$3\ 681 \times 10^{-4}$	$2\ 436 \times 10^{-3}$
1 00	$1\ 931 \times 10^{-4}$	$1\ 265 \times 10^{-3}$
1 10	$1\ 473 \times 10^{-4}$	$9\ 595 \times 10^{-4}$
1 20	$1\ 150 \times 10^{-4}$	$7\ 463 \times 10^{-4}$
1 50	$6\ 118 \times 10^{-5}$	$3\ 930 \times 10^{-4}$
2 0	$2\ 712 \times 10^{-5}$	$1\ 718 \times 10^{-4}$

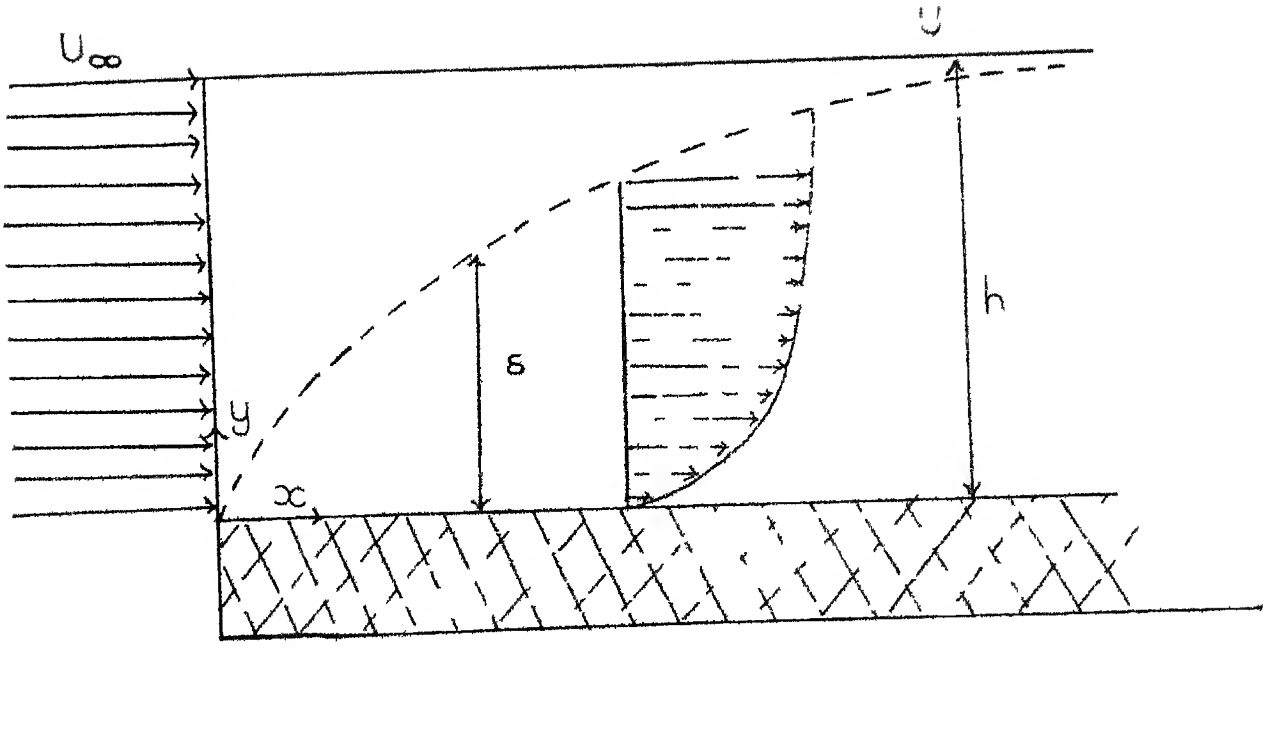

FIG 41 ENTRY FLOW IN A POROUS WALLED CHANNEL

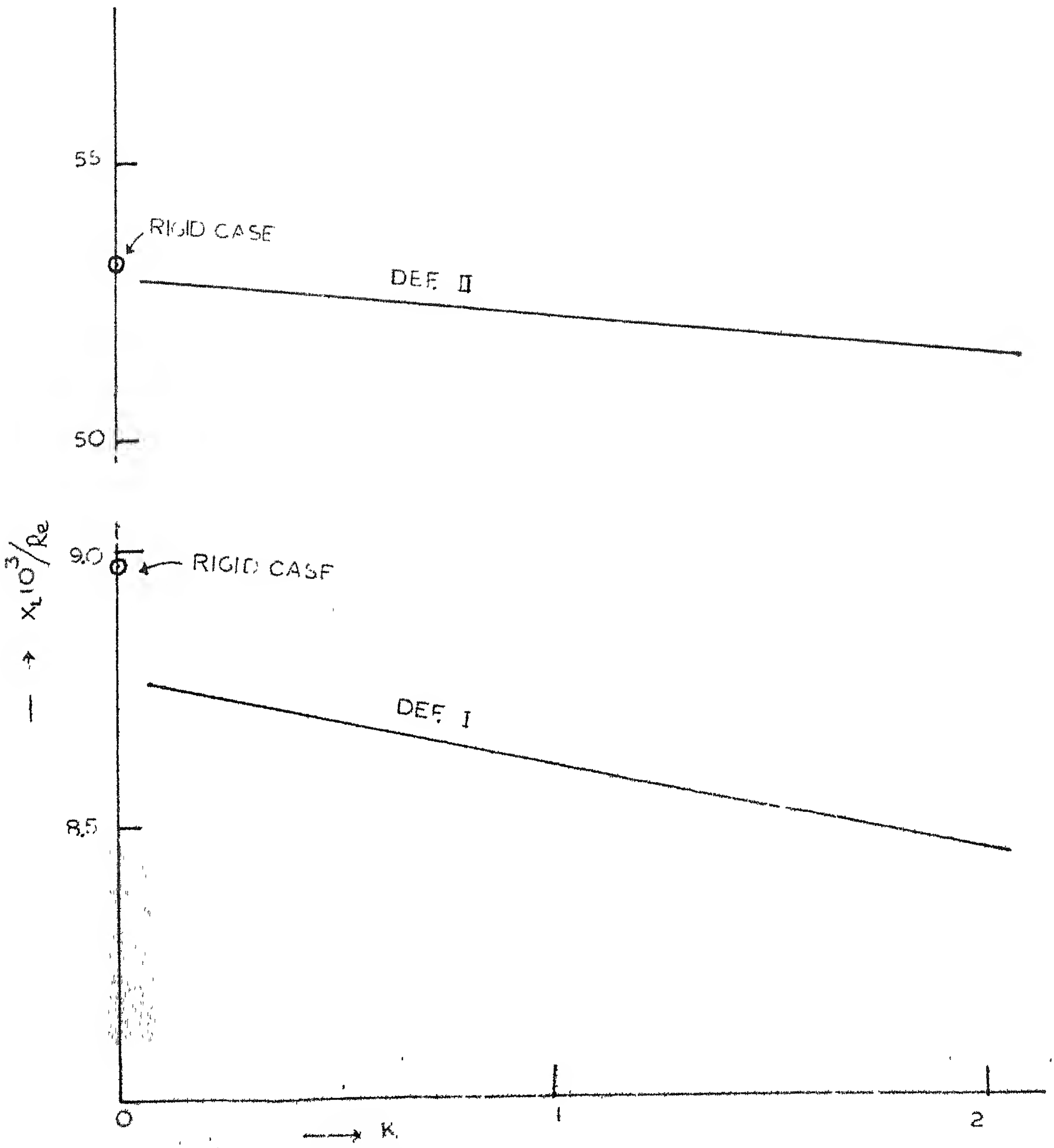

FIG. 4.2 ENTRY LENGTHS: SIMILAR PROFILES

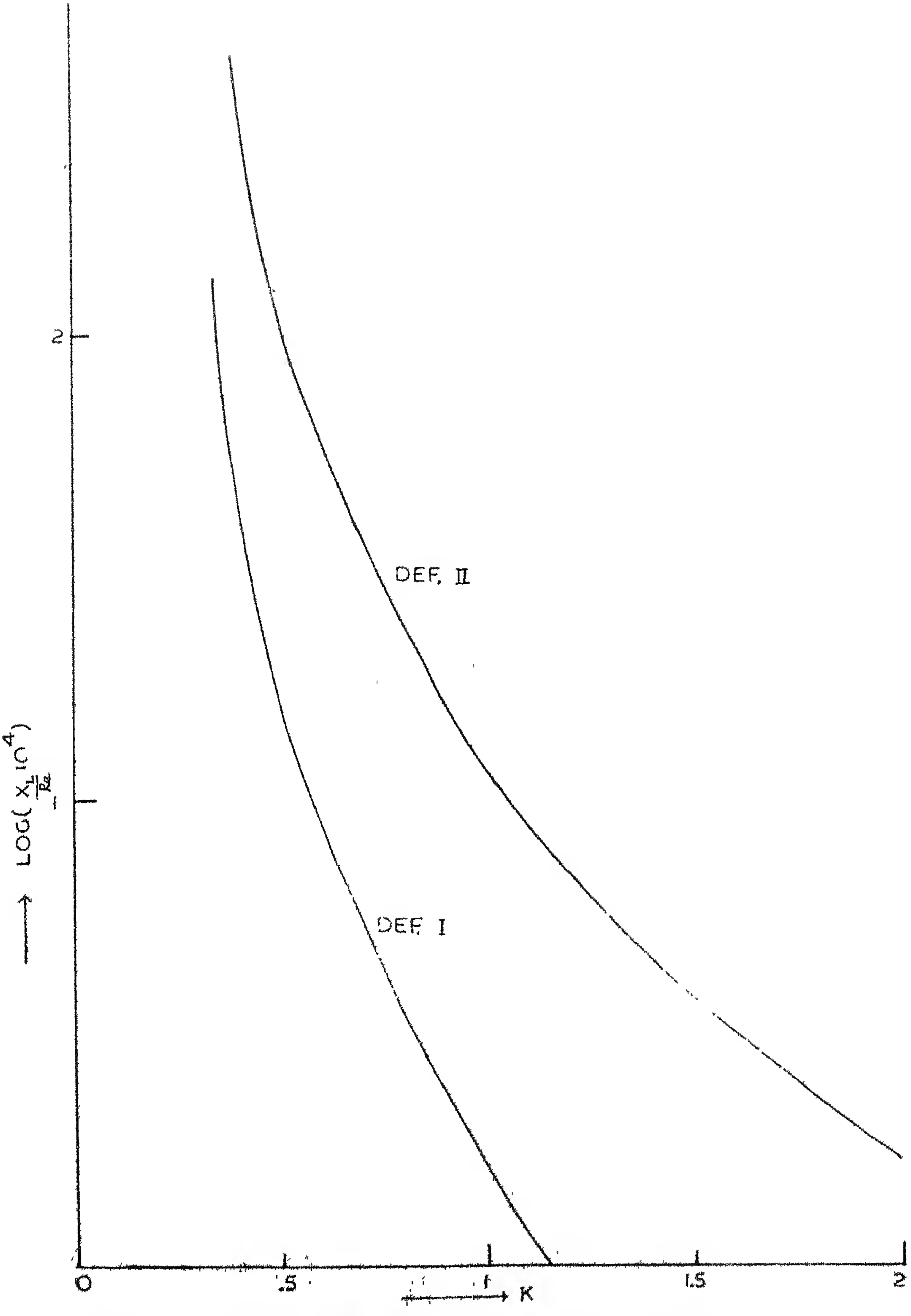

FIG. 4.3 ENTRY LENGTHS: NONSIMILAR PROFILES

CHAPTER - 5

BÉNARD PROBLEM WITH A POROUS BOUNDARY

5.1 Introduction

The classical Bénard problem - the setting up of convection currents in a thin film of an incompressible viscous fluid layer originally at rest due to an adverse temperature gradient across it - has been subject to a thorough study by a number of authors (Chandrasekhar (1961)). The basic mechanism for the instability of the system has been clearly explained by Pearson J. R. A. (1958) in terms of surface tension forces while a similar problem of the breakdown of a fluid subject to a vertical temperature gradient in a homogeneous porous medium and the ensuing convective flow has been studied by Lapwood (1948). Lapwood, using the linear stability theory and the Boussinesq approximation arrives at a fourth order differential equation for the temperature perturbation θ. As regards to the boundary conditions at the upper and lower surfaces of the porous block, the temperature has to satisfy one condition at each surface. But since the solution of the fourth order differential equation admits four degrees of freedom, the velocity can be made to obey one condition only at each boundary which implies that allowance for a state of no-slip cannot be made as this would require two conditions. Lapwood suggests then that the solution of the differential

equation is valid at best upto the neighbourhood of the boundary and ceases to apply in a boundary layer

It has been customary to assume that the tangential velocity is zero at the surface for a flow over a porous surface Beavers and Joseph (1967) have recently proposed an adhoc 'slip-condition' for a flow of the type as mentioned above

The validity of the model proposed by Beavers and Joseph has been tested experimentally and theoretically for certain flow configurations

The linear stability of a laminar flow in a parallel plate channel, one of whose walls is porous, has been investigated by Sparrow, Beavers, Chen and Lloyd (1973) They use the 'adhoc' boundary condition of Beavers and Joseph The validity of this condition has been established by Taylor (1971), Richardson (1971) in companion papers and by Saffman (1971) Sparrow et al observe that the behavior of the critical Reynolds number depends in a complex way on the non-dimensional permeability Λ of the porous medium which can be regarded as a measure of the extent of the interaction between the flows in the channel and the porous material In certain ranges of Λ, it is observed that the critical Reynolds number is substantially reduced while in others it increases An explanation is advanced in terms of several mechanisms by which the permeable material affects the laminar stability limit of the channel flow

At places where the curve shows a decreasing tendency, it is stated that the non-vanishing disturbance velocity is believed to dominate while in the range in which the curve shows an increasing tendency, the effect of the skewness of the main velocity profile dominates

The main purpose of the present investigation is to understand this explanation more clearly If the main velocity distribution is totally absent, then how the non-vanishing of the disturbance velocities is going to affect the stability criterion ? Will it still be destabilizing for all the values of the parameter Λ ? An answer is sought for these questions

5 2 The Physical Problem and the Basic Equations

The model consists of a rectangular porous block of thickness h whose underside is a rigid plate The porous medium is assumed to be both homogeneous and isotropic There is a thin layer of an incompressible viscous fluid of thickness d on top of the porous material which is saturated with the same fluid By heating the underside of the porous material, an adverse temperature gradient is maintained across the system Because of thermal conduction the density of the fluid in the porous medium as well as that of the fluid layer changes This will lead to an instability whenever the adverse temperature gradient exceeds a certain value A sketch of the model is an shown in Fig 5 1

The governing equations of motion for the fluid are the usual Navier-Stokes equations and the heat conduction equation For the fluid in the porous medium, they are the unsteady form of the Darcy's law and the modified heat conduction equation (Katto and Masuoka, 1967) In cartesian tensor notation they are

For fluid

$$\rho \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] = - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} - \rho g \lambda_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa_f \frac{\partial^2 T}{\partial x_j^2}$$

For fluid in permeable medium (5 1)

$$\frac{\rho^*}{\varepsilon} \frac{\partial u_i^*}{\partial t} = - \frac{\partial p^*}{\partial x_i} - \frac{\mu}{\Lambda^*} u_i^* - \frac{\rho^*}{\varepsilon} g \lambda_i$$

$$\frac{\partial u_i^*}{\partial x_i} = 0$$

$$\frac{\partial T^*}{\partial t} + \frac{(c_p \rho)_f}{(c_p \rho)_m} u_j^* \frac{\partial T^*}{\partial x_j} = \kappa_m \frac{\partial^2 T^*}{\partial x_j^2}$$

$$\lambda \equiv (0,0,1)$$

The inertial terms in the Darcy's law have been neglected as they are very small compared to the other terms. Here ε is the porosity of the material, Λ^* the permeability, κ the coefficient of thermal ~~conductivity~~ diffusivity and c_p the specific heat at constant pressure. The subscripts f and m refer to the fluid medium and the porous material saturated with the fluid.

The equations of state have to be supplemented to this set of equations and they are

$$\begin{aligned} \rho &= \rho_1 \,[\,1 - \alpha(T - T_1)\,] \\ \rho^* &= \rho_0 \,[\,1 - \alpha_1(T^* - T_0)\,] \end{aligned} \tag{5 2}$$

where α, α_1 are the coefficients of volume expansion of the fluid and of the fluid in the porous material, T_1 is the temperature at the interface.

5 3 The Perturbation Equations

The initial state when there are no motions is governed by

$$u_i = 0 \qquad p = -g\rho_1 [\,z + \frac{\alpha}{2}\beta_f (z-h)^2\,] + \text{const}$$

$$T = T_1 - \beta_f\,(z - h) = T_0 - \beta_m h - \beta_f(z - h) \tag{5 3}$$

$$u_i^* = 0 \qquad p^* = -\frac{g\rho_0}{\varepsilon}[\,z + \frac{\alpha_1\beta_m}{2} z^2\,] + \text{const}$$

$$T^* = T_0 - \beta_m z$$

where T_o is the temperature at $z = 0$. One notes that the temperature is continuous at the interface $z = h$. The values of β_m and β_f are right now unknown. But if the temperature T_2 at $z = h + d$ is prescribed and the temperature flux be continuous at the interface $z = h$ then one obtains that

$$[\bar{\kappa}_f \frac{\partial T}{\partial z} = \bar{\kappa}_m \frac{\partial T^*}{\partial z} \quad \text{at} \quad z = h]$$

$$T_2 = T_o - \beta_m h - \beta_f d \qquad (5.4)$$

$$\bar{\kappa}_f \beta_f = \bar{\kappa}_m \beta_m \ ,$$ $\bar{\kappa}_f, \bar{\kappa}_m$ are Thermal conductivities

These equations determine β_f and β_m as

$$\beta_f = \frac{\bar{\kappa}_f (T_o - T_2)}{(h\bar{\kappa}_f + d\,\bar{\kappa}_m)}$$

$$\beta_m = \frac{\bar{\kappa}_m (T_o - T_2)}{(h\bar{\kappa}_f + d\,\bar{\kappa}_m)} \qquad (5.5)$$

Also the density at the interface is to be continuous so that

$$\rho_1 = \rho_o [1 - \alpha_1(T_1 - T_o)]$$

i.e. $$\rho_1 = \rho_o (1 + \alpha_1 \beta_m h) \qquad (5.6)$$

Under the Boussinesq approximation $\rho_1 \simeq \rho_o$. Thus there is a fluid of density ρ_o saturating the permeable medium and lying

over making a layer of thickness d The system now is heated from below maintaining the top and the bottom walls at temperatures T_2 and T_o ($> T_2$) Through conduction the density gets modified The purpose now is to investigate the onset of convection in the fluid layer

The flow variables are now perturbed in the usual fashion The linearized perturbation equations under the Boussinesq approximation are

For fluid

$$\rho_o \frac{\partial u_i}{\partial t} = - \frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \alpha g \rho_o \theta \lambda_i + \alpha \alpha_1 \rho_o g \beta_m h \theta \lambda_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial \theta}{\partial t} - \beta_f w = \kappa_f \frac{\partial^2 \theta}{\partial x_j^2}$$

(7)

For permeable Medium

$$\frac{\rho_o}{\varepsilon} \frac{\partial u_i'}{\partial t} = - \frac{\partial P'}{\partial x_i} - \frac{\mu}{\Lambda^*} u_i' + g \frac{\rho_o}{\varepsilon} \alpha_1 \lambda_i \theta'$$

$$\frac{\partial u_i'}{\partial x_i} = 0$$

$$\frac{\partial \theta'}{\partial t} - \gamma \beta_m w' = \kappa_m \frac{\partial^2 \theta'}{\partial x_j^2}$$

where $\gamma = \frac{(c_p \rho)_f}{(c_p \rho)_m}$.

The last term of the first equation of the set (5 7) can be dropped as there occurs the product of α and α_1 which are both small

The physical condition at the upper rigid plate adjacent to the fluid, requires that the velocity be zero However at the lower rigid plate adjacent to the porous material (i e at $z = 0$) only normal component of velocity can vanish At the interface $z = h$, transverse velocity, temperature, normal stresses and temperature flux are to be continuous Further to include the interaction between the fluid and the porous medium, the conditions proposed by Beavers and Joseph are made use of Then the boundary conditions can be written as

$$
\begin{aligned}
&\text{At} \quad z = 0 \qquad && \theta' = 0 = w' \\
&\text{At} \quad z = d+h \qquad && \theta = 0 = w = \frac{\partial w}{\partial z} \qquad\qquad (5\ 8) \\
&\text{At} \quad z = h \qquad && w = w', \theta = \theta', \ \bar{\kappa}_m \frac{\partial \theta'}{\partial z} = \bar{\kappa}_f \frac{\partial \theta}{\partial z}, \\
& && -p' = -p + \mu \frac{\partial w}{\partial z} \\
& && \frac{\partial}{\partial z}(u,v) = \frac{\alpha_o}{\sqrt{\Lambda^*}}[(\hat{u},\hat{v}) - (u', v')]
\end{aligned}
$$

Here u,v and $\hat{u}'$, $\hat{v}'$ are the velocities of the fluid and the velocities of the fluid in the porous medium, α_o is the slip coefficient $\hat{u}'$, $\hat{v}'$ are to be determined from the Darcy's law

Now using the normal mode analysis the perturbed flow variables are expanded as

$$f(x,y,z,t) = f(z) \exp\left[ik_x x + ik_y y + pt\right] \tag{5.9}$$

where p is complex in general. Making the transformations

$$\bar{z} = \frac{z}{d}, \quad a = kd, \quad \sigma = \frac{pd^2}{\nu}, \quad P_m = \frac{\nu}{\kappa_m}, \quad P_f = \frac{\nu}{\kappa_f},$$

$$\Lambda = \frac{\Lambda^*}{\varepsilon d^2}, \quad D \equiv \frac{d}{dz}, \quad k^2 = k_x^2 + k_y^2, \quad \nu = \frac{\mu}{\rho_0} \tag{5.10}$$

the perturbation equations after simplifications and eliminations reduce to

$$(D^2 - a^2)(D^2 - a^2 - \sigma)\, w = \frac{R_f \kappa_f}{\beta_f d^2}\, a^2 \theta$$

$$\frac{h}{d} < z \leq 1 + \frac{h}{d} \tag{5.11}$$

$$(D^2 - a^2 - \sigma P_f)\theta = -\frac{\beta_f}{\kappa_f}\, d^2 w$$

$$(\sigma\Lambda + 1)(D^2 - a^2) w' = -\frac{R_m \kappa_m \Lambda a^2 \theta'}{\gamma \beta_m d^2}$$

$$0 \leq z \leq \frac{h}{d} \tag{5.12}$$

$$(D^2 - a^2 - \sigma P_m)\, \theta' = -\frac{\gamma \beta_m d^2}{\kappa_m}\, w'$$

where $R_f = \dfrac{g \alpha \beta_f}{\kappa_f \nu} d^4, \quad R_m = \dfrac{\gamma g \alpha_1 \beta_m}{\kappa_m \nu} d^4$

Further, perturbations inpressure within fluid and porous medium will be given by the following

$$P = \frac{\mu}{a^2 d}(D^2 - a^2 - \sigma)\, DW$$

$$P' = \frac{\mu}{\varepsilon \Lambda d a^2}(1 + \sigma\Lambda)\, DW'$$

Making use of the above expressions for P and P', the boundary conditions (5 8) come out to be

At $z = 0$ $\qquad \theta' = w' = 0$

At $z = 1 + \frac{h}{d}$ $\qquad \theta = w = 0 = Dw$ (5 13)

At $z = \frac{h}{d}$ $\qquad \theta = \theta', \; w = w', \; \bar{\kappa}_m D\theta' = \bar{\kappa}_f D\theta$

$$(D^2 - 2a^2 - \sigma)Dw + \frac{1}{\varepsilon\Lambda}(1+\sigma\Lambda)\,Dw' = 0$$

$$D^2 w = \frac{\alpha_o}{\sqrt{\varepsilon\Lambda}}(Dw - Dw')$$

5 4 Evaluation of the Critical Rayleigh Numbers

The system of equations (5 14), (5 15) along with the set (5.13) form eigen value problems for the Rayleigh numbers R_f and R_m The critical value of these numbers will be determined by the use of Galerkin's method

In the following discussion the principle of exchange of stabilities is assumed to hold So much so putting $\sigma = 0$, the set of Equations (5 11) and (5 12) becomes

$$(D^2 - a^2)^2 = \frac{R_f \kappa_f}{\beta_f d^2}\, a^2 \theta$$

$$\frac{h}{d} < z \le 1 + \frac{h}{d} \qquad (5\ 14)$$

$$(D^2 - a^2)\,\theta = -\frac{\beta_f}{\kappa_f}\, d^2 w$$

$$(D^2 - a^2)w' = -\frac{R_m \kappa_m\, \Lambda a^2 \theta'}{\gamma\beta_m d^2} \qquad 0 \le z \le \frac{h}{d}$$

(5 15)

$$(D^2 - a^2)\theta' = -\frac{\gamma\beta_m\, d^2}{\kappa_m}\, w'$$

The corresponding boundary conditions are

$$\text{At} \quad z = 0 \qquad \theta' = w' = 0$$

$$\text{At} \quad z = 1 + \frac{h}{d} \qquad \theta = w = 0 = Dw$$

$$\text{At} \quad z = \frac{h}{d} \qquad \theta = \theta',\ w = w',\ \bar{\kappa}_m D\theta' = \bar{\kappa}_f D\theta \tag{5 16}$$

$$(D^2 - 2a^2)Dw + \frac{1}{\varepsilon\Lambda} Dw' = 0$$

$$D^2 w = \frac{\alpha_0}{\sqrt{\varepsilon\Lambda}} (Dw - Dw')$$

In order to apply the above method, the following four functions are defined

$$w = (z - 1 - \frac{h}{d})^2 (a_0 + a_1 z + z^2)$$

$$\theta = (z - 1 - \frac{h}{d}) (b_0 + z) \tag{5 17}$$

$$w' = (c_0 + z) z$$

$$\theta' = z(d_0 + z)$$

These are the 'test' functions and satisfy part of the boundary conditions The constants a_0, a_1, b_0, c_0 and d_0 are determined by the remaining conditions Their values are

$$a_1 = (N_1 N_2 - N_3 N_4)/(N_5 N_1 + N_3 N_6)$$

$$a_0 = (N_2 - a_1 N_5)/N_3$$

$$c_0 = a_1 + a_0\ d/h \tag{5 18}$$

$$b_0 = [\frac{h}{d} (1 - \frac{h}{d})(1 - \frac{\bar{\kappa}_m}{\bar{\kappa}_f})] / (\frac{h}{d} + \frac{\bar{\kappa}_m}{\bar{\kappa}_m})$$

$$d_0 = - [1 + \frac{h^2}{d^2} + 2 \frac{h}{d} \frac{\bar{\kappa}_m}{\bar{\kappa}_m}] / (\frac{h}{d} + \frac{\bar{\kappa}_m}{\bar{\kappa}_f})$$

where
$$N_1 = \frac{\alpha_o}{\sqrt{\varepsilon\Lambda}} + \frac{2h}{d}\left(1 + \frac{\alpha_o}{\sqrt{\varepsilon\Lambda}}\right)$$
$$N_2 = 12\frac{h}{d} - 4a^2\frac{h^3}{d^3} + \frac{h^2}{d^2}\left(4a^2 - 12 - \frac{2}{\varepsilon\Lambda}\right)$$
$$N_3 = \left(\frac{1}{\varepsilon\Lambda} + 4a^2\frac{h}{d}\right)$$
$$N_4 = -\frac{2h}{d} + \frac{8h^2}{d^2} - \frac{2h^3}{d^3}\left(1 + \frac{\alpha_o}{\sqrt{\varepsilon\Lambda}}\right)$$
$$N_5 = \frac{h}{d}\left(6 + \frac{1}{\varepsilon\Lambda} - 2a^2 + 4a^2\frac{h}{d}\right)$$
$$N_6 = \frac{4h}{d} - \frac{2h^2}{d^2}\left(1 + \frac{\alpha_o}{\sqrt{\varepsilon\Lambda}}\right)$$

Let now

$$w = \sum_{n=1}^{N} A_n \left(z - 1 - \frac{h}{d}\right)^{2n} \left(a_o + a_1 z + z^2\right)^n$$

$$\theta = \sum_{n=1}^{N} B_n \left(b_o + z\right)^n \left(z - 1 - \frac{h}{d}\right)^n \tag{5 19}$$

$$w' = \sum_{n=1}^{N} C_n \left(c_o + z\right)^n z^n$$

$$\theta' = \sum_{n=1}^{N} D_n z^n \left(d_o + z\right)^n$$

As is done in the Galerkin's method, these expression are substituted in the set of Equations (5 14) and (5 15) Since these are not the actual solutions, these will not satisfy the differential equations (5 14) and (5 15) So much so the substitution of these expressions into the differential equations will result in residuals E_r, $P(w) - Q(\theta) = E_r$ where P,Q are polynomials in z coming from terms

depending on w and θ respectively The residuals corresponding to each of the equations of the set (5 14) and (5 15) are normalized This will lead to a set of four homogeneous linear equations in the coefficients A_1, B_1, C_1, D_1 In order that a non-trivial solution exists for these coefficients, an infinite ordered determinant has to vanish By sucessively taking the first, second etc terms in the expansion, one can determine the critical Rayleigh numbers Infact the Rayleigh numbers will satisfy functional relations of the form

$$R_f = f\left(a_o, a_1, b_o, c_o, \alpha_o, \Lambda, \frac{h}{d}, a\right)$$
$$R_m = g\left(a_o, a_1, c_o, d_o, \alpha_o, \Lambda, \frac{h}{d}, a\right) \qquad (5.20)$$

From these the critical Rayleigh numbers can be determined

The evaluations were done taking the first and the first two terms of the expansions The calculations are indicated in the following The use of the procedure outlined in the previous paragraph will result in the following system of equations for the case when the first terms are used

$$(24S(1) - 2a^2 V^*_{11} + a^4 V^*_{21})\, A_1 - \frac{R_f \kappa_f}{\beta_f d^2} a^2 V^*_{31} B_1 = 0$$
$$\frac{\beta_f d^2}{\kappa_f} V^*_{31} A_1 + (2U(1) - a^2 V^*_{41})\, B_1 = 0 \qquad (5\ 21)$$

$$[c_o(\tfrac{h}{d})^2 + \tfrac{2}{3}(\tfrac{h}{d})^3 - a^2\{\frac{c_o^2}{3}(\tfrac{h}{d})^3 + \tfrac{1}{2}c_o(\tfrac{h}{d})^4 + \tfrac{1}{5}(\tfrac{h}{d})^5\}]\quad C_1$$
$$+ \frac{R_m \kappa_m \Lambda a^2}{\gamma \beta_m d^2}\left[\frac{c_o d_o}{3}(\tfrac{h}{d})^3 + \frac{c_o + d_o}{4}(\tfrac{h}{d})^4 + \tfrac{1}{5}(\tfrac{h}{d})^5\right] D_1 = 0 \qquad (5\ 22)$$

$$\frac{\gamma\beta_m d^2}{\kappa_m}\left[\frac{c_o d_o}{3}\left(\frac{h}{d}\right)^3 + \frac{c_o + d_o}{4}\left(\frac{h}{d}\right)^4 + \frac{1}{5}\left(\frac{h}{d}\right)^5\right] C_1$$

$$+ \left[d_o\left(\frac{h}{d}\right)^2 + \frac{2}{3}\left(\frac{h}{d}\right)^3 - a^2\left\{\frac{d_o^2}{3}\left(\frac{h}{d}\right)^3 + \frac{1}{2} d_o\left(\frac{h}{d}\right)^4 + \frac{1}{5}\left(\frac{h}{d}\right)^5\right\}\right] D_1 = 0$$

where

$$V_{11}^* = Q_{11}S(1) + Q_{12}S(2) + Q_{13}S(3)$$

$$V_{21}^* = P_{11}S(1) + P_{12}S(2) + P_{13}S(3) + P_{14}(4)\ S(4) + S(5)$$

$$V_{31}^{t} = R_{11}\ (1) + R_{12}S(2) + S(3)$$

$$V_{41}^* = R_{11}U(1) + R_{12}U(2) + U(3)$$

$$R_{11} = \left(b_o + \frac{h}{d}\right) ;\ R_{12} - b_o + \frac{h}{d} \quad 1$$

$$P_{11} = a_o + a_1\frac{h}{d} + \frac{h^2}{d^2} \quad P_{12} - \quad 1 + \frac{2h}{d} \quad 2P_{11}$$

$$P_{13} - 1 - 2\left(a_1 + \frac{2h}{d}\right) + P_{11}$$

$$P_{14} = a_1 + \frac{2h}{d} \quad 2$$

$$Q_{11} = 2\ P_{13};\ Q_{12} - 6P_{14} \quad Q_{13} = 12$$

$$S(i) = \int_0^1 \left(P_{11} + P_{12}z + P_{13}z^2 + P_{14}\ {}^3 + z^4\right)\ {}^{i-1}d \quad i = 1\ 2 \quad ,9$$

$$U(i) = \int_0^1 \left(R_{11} + R_{12}z + z^2\right) z^{i\ 1}\ dz,\ i = 1,2 \quad ,9$$

where a_o, a_1, b_o, c_o, d_o ar as given in Equation (5 18)

From Equations (5 21) and (5 22) one btain

$$R_f = -\,(24S(1) - 2a^2V^*_{11} + a^4V^*_{21})\ (2U(1) - a^2V^*_{41})/a^2V^{*2}_{31} \qquad (5\ 23)$$

$$R_m = [\,(c_o + \frac{2}{3}\frac{h}{d}) - a^2\ \{\frac{c_o^2}{3}\ (\frac{h}{d})^2 + \frac{1}{2}\ c_o(\frac{h}{d})^2 + \frac{1}{5}\ (\frac{h}{d})^3\}]\ \text{times}$$

$$[\,(d_o + \frac{2}{3}\frac{h}{d}) - a^2\ \{\frac{d_o^2}{3}\ (\frac{h}{d}) + \frac{1}{2}\ d_o(\frac{h}{d})^2 + \frac{1}{5}\ (\frac{h}{d})^3\}]\ \text{divided by}$$

$$\Lambda a^2(\frac{h}{d})^2[\ \frac{c_o d_o}{3} + \frac{c_o + d_o}{4}\ \frac{h}{d} + \frac{1}{5}\ \frac{h^2}{d^2}\ ]^2$$

When the first two terms in the expansions are used, the following equation is got for the Rayleigh number R_f

$$R_f = (-V32 \pm \sqrt{(V32)^2 - 4\ V31\ V33})\ /\ 2\ V33 \qquad (5\ 24)$$

Here

$$V31 = (V25\ V16\ - V15\ V26)\ (V11\ V22 - V12\ V21)$$

$$V32 = -[(V21\ V28 - V27\ V22)\ (V13\,V16\ - V15\ V14)$$

$$+ (V21\ V18 - V17\ V22)\ (V25\ V14\ - V13\ V26)$$

$$+ (V27\ V12 - V11\ V28)\ (V23\ V16\ - V15\ V24)$$

$$+ (V11\ V18 - V17\ V12)\ (V23\ V26\ - V24\ V25)]\ a^2$$

$$V33 = (V23\ V14\ - V13\ V24)\ (V17\ V28\ - V27\ V18)\ \ a^4$$

$$V11 = 24\ S(1) - 2a^2\ \{Q11\ S(1) + Q12\ S(2) + Q13\ S(3)\}$$

$$+ a^4\ \{P_{11}S(1) + P_{12}S(2) + P_{13}S(3) + P_{14}S(4) + S(5)\}$$

$$V12 = 24\ \{S13\ S(1) + 5\ S12\ S(2) + 15\ S11\ S(3) - 70\ P14\ \ S(4)$$

$$+ 70\ S(5)\} - 4a^2\ \{S15\ S(1) + 3\ S14\ S(2) + 6\ S13\ S(3)$$

$$+ 10\ S12\ S(4) + 15\ S11\ S(5) + 42\ P14\ S(6) + 28\ S(7) \}$$

$$+ a^4 \{P17\ S(1) + S16\ S(2) + S15\ S(3) + S14\ S(4) + S13\ S(5)$$

$$+ S12\ S(6) + S11\ S(7) + S18\ S(8) + S(9) \}$$

$$V13 = R11\ S(1) + R12\ S(2) + S(3)$$

$$V14 = R13\ S(1) + R14\ S(2) + R15\ S(3) + R16\ S(4) + S(5)$$

$$V15 = 2U(1) - a^2 \{R11\ U(1) + R12\ U(2) + U(3)\}$$

$$V16 = R17\ U(1) + R18\ U(2) + R19\ U(3) - a^2\{R13\ U(1) + R14\ U(2)$$

$$+ R15\ U(3) + R16\ U(4) + U(5)\} \qquad (5\ 25)$$

$$V17 = P11\ U(1) + P12\ U(2) + P13\ U(3) + P14\ U(4) + U(5)$$

$$V18 = P17\ U(1) + S16\ U(2) + S15\ U(3) + S14\ U(4)$$

$$+ S13\ U(5) + S12\ U(6) + S11\ U(7) + S18\ U(8) + U(9)$$

V21, V22, V23, V24 are got by replacing $S(i)$ by $T(i)$ in V11, V12, V13, V14 and V25, V26, V27, V28 are got by replacing $U(i)$ by $R(i)$ in Equation (5 25) The expression for $T(i)$ and $R(i)$ are

$$T(i) = \int_0^1 [P17 + S16\ z + S15\ z^2 + S14\ z^3 + S13\ z^4 + Z12\ z^5$$

$$+ S11\ z^6 + S18\ z^7 + z^8]\ z^{i-1}\ dz\ ,\ i = 1, \quad ,9,$$

$$R(i) = \int_0^1 (R13 + R14z + R15\ z^2 + R16\ z^3 + z^4)\ z^{i-1}\ dz,\ i=1, \quad ,9$$

where

$$S11 = (6 - 8\ P15 + P16 + 2P11)$$

$$S12 = 2(-2 + 6\ P15 - 2P16 + P18 - 4P11)$$

$$S13 = 1 - 8\,P15 + 6\,P16 - 8\,P18 + 12\,P11 + P17$$

$$S14 = 2(P15 - 2\,P16 + 6\,P18 - 4\,P11 - 2\,P17)$$

$$S15 = P16 - 8\,P18 + 2\,P11 + 6\,P17)$$

$$S16 = 2(P18 - 2\,P17)$$

$$S17 = 1$$

$$S18 = 2\,P14,$$

$$P15 = a_1 + 2h/d$$

$$P16 = (a_1 + 2h/d)^2$$

$$P17 = (a_o + a_1\frac{h}{d} + \frac{h^2}{d^2})^2$$

$$P18 = (a_1 + \frac{2h}{d})\,(a_o + a_1\,\frac{h}{d} + \frac{h^2}{d^2})$$

$$R13 = (b_o + h/d)^2$$

$$R14 = 2(b_o + h/d) - 2(b_o + \frac{h}{d})^2$$

$$R15 = (b_o + \frac{h}{d})^2 - 4(b_o + \frac{h}{d}) + 1$$

$$R16 = 2(b_o + \frac{h}{d}) - 2$$

$$R17 = 2\,R15$$

$$R18 = 6\,R16$$

$$R19 = 12$$

The remaining variables have already been defined

The critical Rayleigh numbers R_f and R_m corresponding to the fluid and the fluid in the porous medium have been calculated numerically for different values of the parameters α_o Λ, $\bar{\kappa}_m/\bar{\kappa}_f$ from Equations (5 23)and (5 24) They have been graphically shown in Fig 5.2, 5 3 and 5 4 A representative sample of these values are also given in Tables (5 1), (5 2), and (5 3)

Discussion

The critical Rayleigh numbers were determined for different values of the parameters The values of the parameters were chosen to be 0 1, 0 12, 0 13, 0 14, 0 15, 0.16 for α_0, 2000, 1000, 600, 500, 400, 300, 250, 200, 135, 100 for $1/\sqrt{\Lambda}$ and 0 1, 0 5, 8, 1 0, 1 5, 2 0 for $\bar{\kappa}_m/\bar{\kappa}_f$ The calculations were also performed for different values of $\frac{h}{d}$ ranging from 0 1 to 11 0 The critical values of the Rayleigh number R_f are shown in Figs 5 2, 5 3 and that of R_m in Fig 5 4 for some of the above values of the parameters

From the figures one observes the following For R_f, the critical Rayleigh numbers continuously increase with increasing values of $1/\sqrt{\Lambda}$ justifying the explanation of Sparrow et al For R_m, the critical values increase quite rapidly with $1/\sqrt{\Lambda}$ while they seem to tend to a constant value for the case of R_f Infact one can easily determine from Equation (5 23) that R_m increases to infinity as Λ increases to zero ($1/\Lambda \to \infty$) whenever the parameter $\bar{\kappa}_m/\bar{\kappa}_f$ is a non-negative finite constant Also the critical R_m's are far greater than those of R_f's which implies that the instability sets in the fluid layer faster than in the medium Also the curves for larger α_0 areabove those for smaller α_0

Qualitatively, the results are as follows for the different values of the parameters The values of the critical numbers are highly dependent on the ratio h/d For example, R_f and R_m were

found to be of the order of 10^{-1} and 10^{5} for h/d = 11 while for h/d = 0 1 these were of the order 10^{3} and 10^{9} respectively For increasing values of h/d, the critical values decrease in a proportionate fashion. When h/d is small, the saturated porous block is quite thin as compared to the fluid layer over it The permeable boundary is probably affecting the stability of the fluid layer only through the slip boundary condition The patterns within the porous material donot probably have much influence on the fluid layer above That explains the closeness of R_f to critical Rayleigh numbers in the Benard problem However, if h/d is large, the flow behavior within the porous material will have significant effect on the stability of the fluid layer which will be very thin compared to the block For large h/d, a more careful treatment within the porous material will be desirable for any conclusive results As the principal aim of the present investigation is to know the effect of the slip condition on Benard problem, more attention has been paid to the case when h/d is small

The results of the 'second iteration' show that the convergence of the Galerkin method used is fairly good One sees that for certain range of $1/\sqrt{\Lambda}$, the critical numbers for the 'first' and 'second' iteration differ only is the unit place One then can anticipate the result to converge in a wider range of $1/\sqrt{\Lambda}$ if more and more terms are taken in the expansion of the Galerkin method.

The critical values of $10^6 \times R_m$ for different α_o's are also close As such only the values for $\alpha_o = 0$ 16 and 0 1 have been given in the Table 5 3

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Table (5 1)

$\frac{h}{d} = 0\ 1$	$\alpha_o = 0\ 10$, $\bar{\kappa}_m / \bar{\kappa}_f = 0\ 8$		$\alpha_c = 0\ 10$, $\bar{\kappa}_m / \bar{\kappa}_f = 0\ 8$	
$\frac{1}{\sqrt{\Lambda}}$	$R^{*}_{f_I}$	$R^{**}_{f_{II}}$	R_{f_I}	$R_{f_{II}}$
2000	1935 3377	1853 8853	1916 7885	1846 3251
1000	1904 7517	1847 3609	1870 7446	1811 8040
600	1866 9587	1809 6172	1816 2640	1773 4874
500	1849 2328	1797 1821	1791 5587	1755 8980
400	1823 9751	1780 6733	1757 2648	1728 3276
300	1785 0571	1749 7215	1706 4192	1686 6319
200	1717 3275	1695 6527	1623 2067	1613 6689
100	1543 6556	1563 7082	1461 7589	1457 3213

* Critical Rayleigh number for first iteration
** Critical Rayleigh number for second iteration

Table (5 2)

$\frac{h}{d} = 0.1$	$\alpha_o = 0.16$	$\bar{\kappa}_m / \bar{\kappa}_f = 0\ 1$	$\alpha_o = 0\ 10$	$\bar{\kappa}_m / \bar{\kappa}_f = 0\ 1$
$\frac{1}{\sqrt{\Lambda}}$	R_{f_I}	$R_{f_{II}}$	R_{f_I}	$R_{f_{II}}$
2000	1899.5383	1823 2170	1875 9276	1820 8809
1000	1860.7847	1816 3834	1818 1626	1799 3785
600	1813 4283	1809.8053	1750 3311	1765 2534
500	1791 3063	1799 3423	1719 7782	1765 9761
400	1759 8815	1788 2004	1677 5523	1721 0298
300	1711 7395	1738 6349	1615 3933	1679 5157
250	1676.5786	1720 7627	1571 8208	1651 1472
200	1628 6540	1709 8504	1514 7806	1618 0927
135	1532 2471	1634 0068	1407 5327	1521 2299
100	1450 5355	1561 4708	1323 6707	1442 3116

Table (5 3)

$\frac{h}{d} = 0.1$	$\alpha_o = 0.16$		$\alpha_o = 0.1$	
	$\bar{\kappa}_m / \bar{\kappa}_f = 0\ 1$	$\bar{\kappa}_m / \bar{\kappa}_f = 0\ 8$	$\bar{\kappa} / \bar{\kappa}_f = 0\ 1$	$\bar{\kappa}_m / \bar{\kappa}_f = 0\ 8$
$\frac{1}{\sqrt{\Lambda}}$	$10^6 \times R_m$	$10^6 \times R_m$	$10^6 \times R_m$	$10^6 \times R_m$
2000	1364.68610	1763 38290	1364 68680	1763 38440
1000	341 19651	440 87664	341 19783	440 87980
600	122 85328	158 74494	122 85536	158 74999
500	85 32599	110 25468	85 32845	110 26062
400	54 62233	70 58195	54 62530	70 58917
300	30 74259	39 72755	30 74637	39 73672
250	21 36193	27 60779	21 36630	27 61839
200	13.68769	17 69368	13 69287	17 70626
135	6 26332	8 10535	6 27009	8 12190
100	3 46178	4 49066	3 46983	4 51061

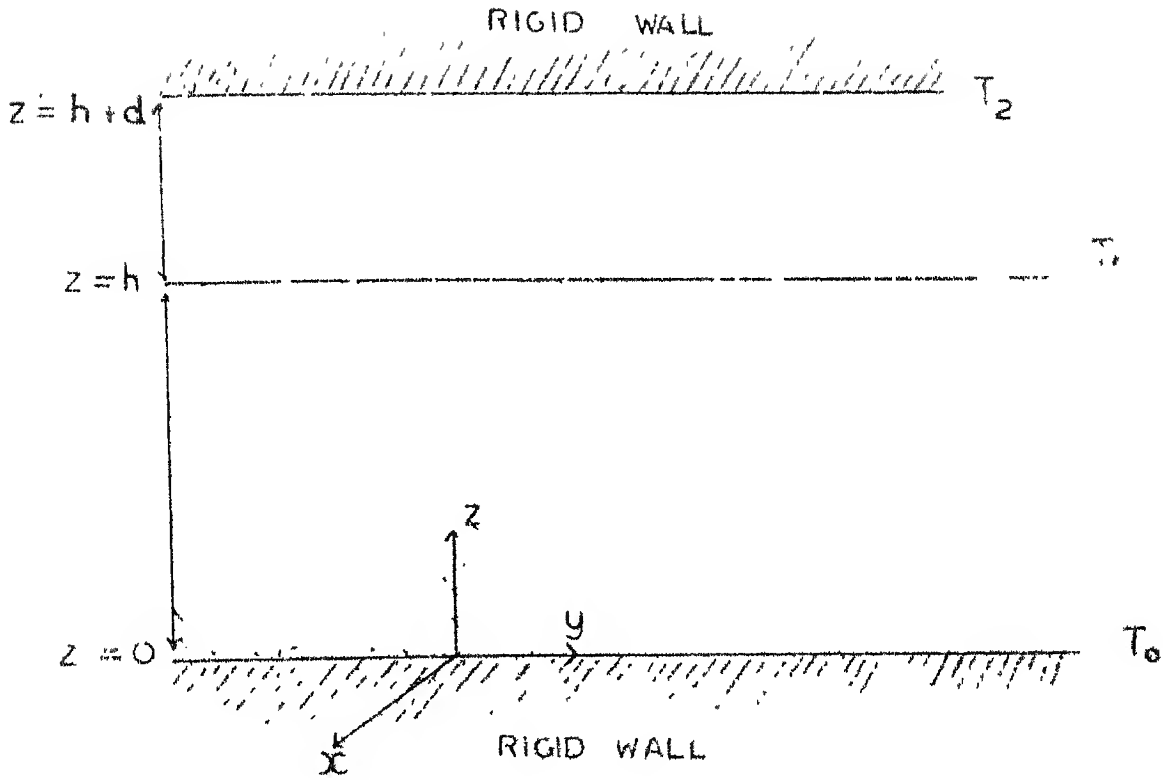

FIG. 5.1 BENARD PROBLEM WITH POROUS BOUNDARY

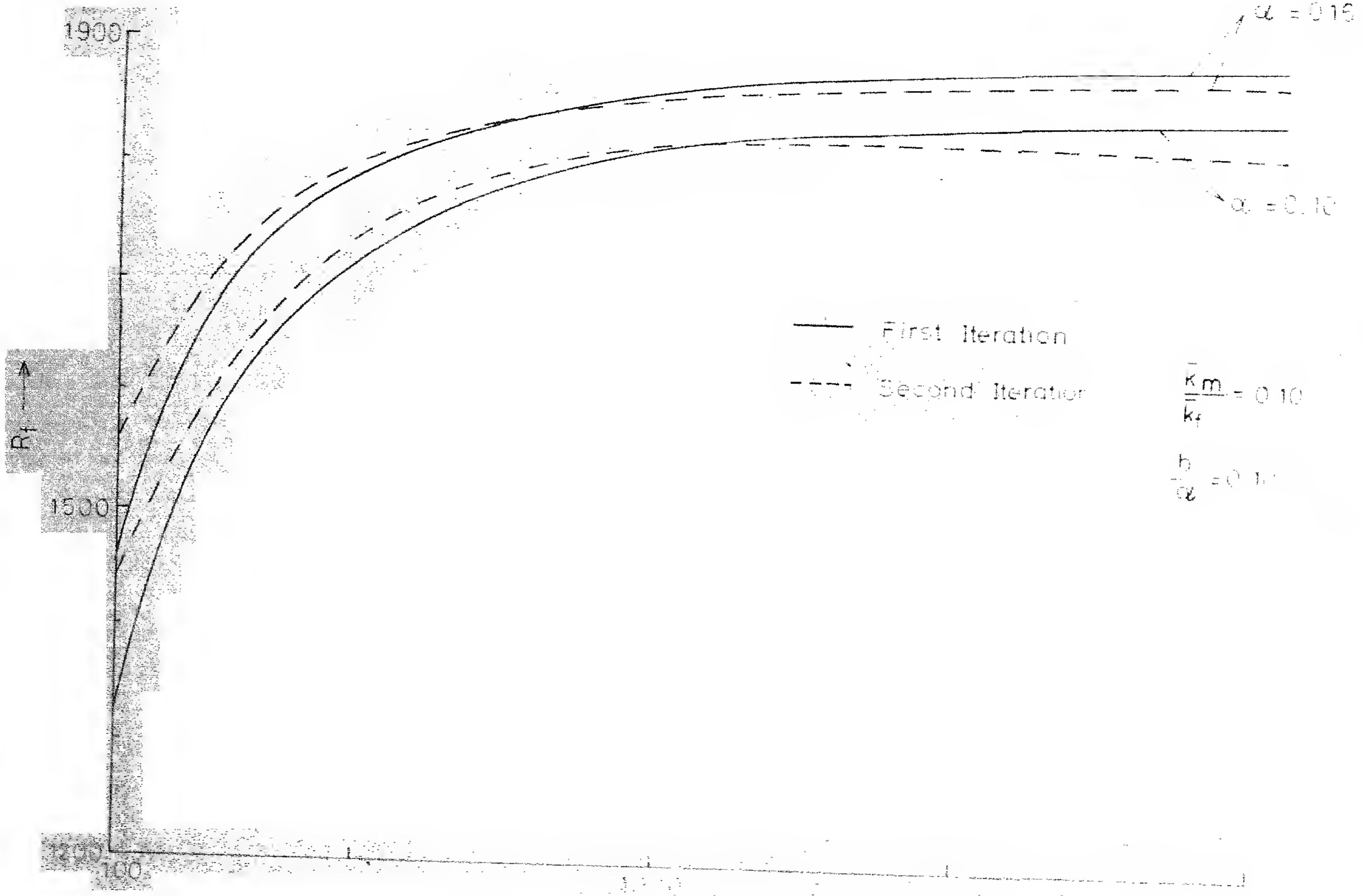

Fig. 5.2 Critical Rayleigh number R_f

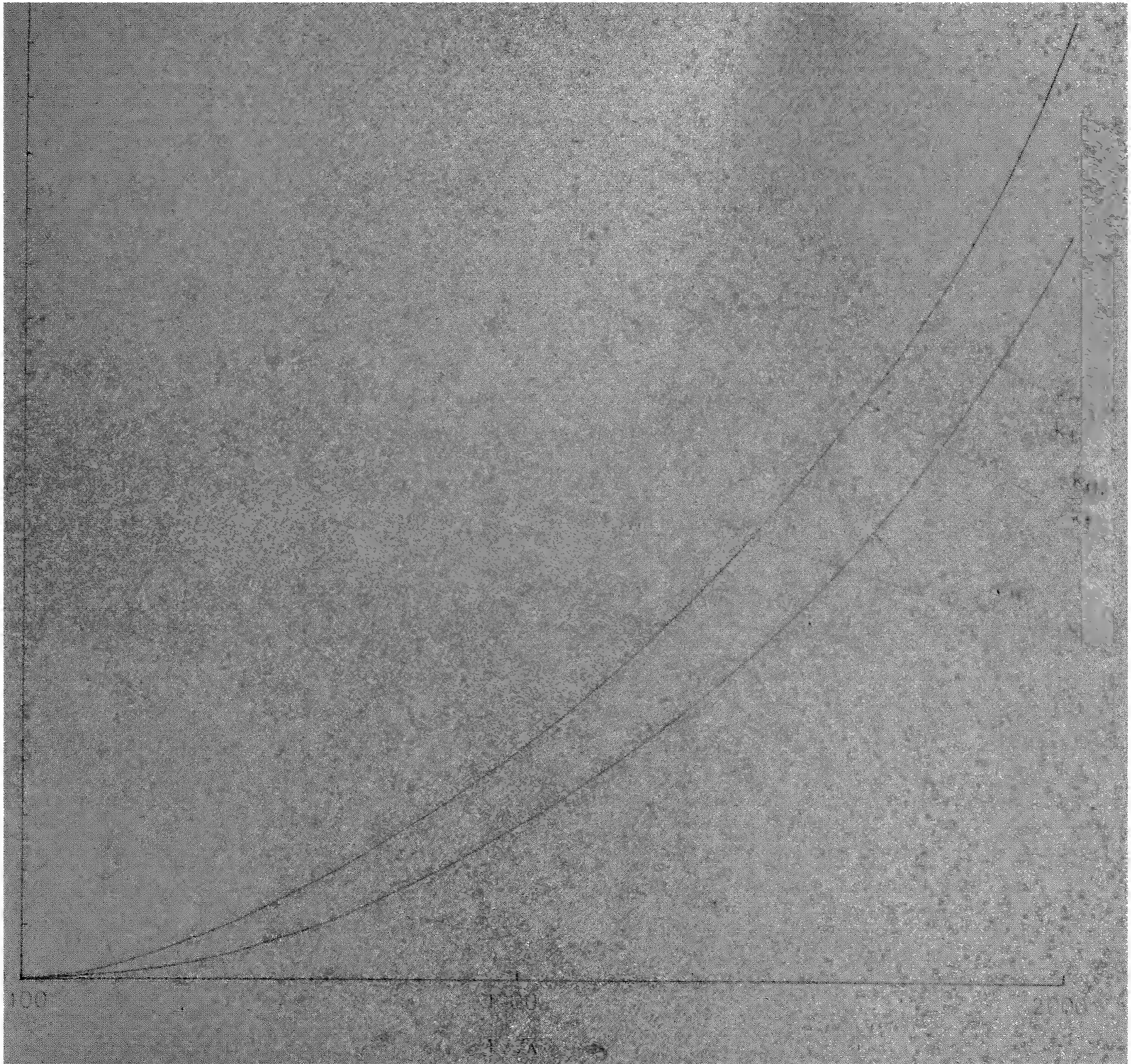

Fig 5.4 Critical Rayleigh number R_m vs $1/\sqrt{A}$

CHAPTER - VI

SOME COMMENTS ON THE THESIS TOPIC

The discussion about the problems dealt with in the thesis has been mostly in a qualitative manner The reason is simple it is because of the non-availability of the exact boundary condition(s) at the interface of a porous material for developing flows The proposed condition is rather intuitive and no rigorous proof for the condition has yet been got Possibly some light would be thrown on the condition with some experiments of developing flows on porous material Once the correct boundary condition(s) is available, a numerical integration of the full Navier-Stokes equations would give a clearer picture of the entry flow in a porous walled channel This would lead to accessory problems like the convergence of the method used, stability, and the computer memory needed Again one has to discuss the existence and uniqueness of the solution for the PDE

Again for the problem of flow over the porous flat plate, no proof of existence and uniqueness of solution has been found as yet A proof on the lines referred to by Nickel (1973) in his review paper may be possible The second solution for the problem has been attempted in this light

For the Benard problem, the principle of exchange of stability has not been proved But for the two mediums - fluid and porous - the proof of such a phenomenon is available of course using the no-slip conditions

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